SUMMER '17 POLISH MODEL THEORY CONFERENCES

Unreliable Notes

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Readme

Disclaimer These notes have been typeset "on the fly" in LaTeX during Wrocław and Będlewo model theory meetings held respectively from the 30^{th} of June to the 2^{nd} of July and from the 3^{rd} to the 7^{th} of July 2017.

They have *not* been reviewed, and were originally intended only for personal use of the author. As a result, they may contain severe errors, huge omissions, and are in general only what I managed to write down. In particular, they are *not* official notes, and have not been reviewed by the speakers.

Because of the reasons above, these notes can be quite embarrassing, if not just pure nonsense. Please forgive me when this happens (it will), especially if this happens with notes from your talk.

Emails pointing out errors, mistakes, etc. are very welcome. In particular, if you spoke in the conference and want to correct these notes, or supplement them, or have them removed, please contact me.

Notation Omissions are marked [like this], sometimes with some words saying what is being omitted, sometimes just with dots as in [...]. Also, if you see [slides] at the beginning of a talk, it means the talk had slides. These should be available on the conference website. Typically, talks with slides will have notes more incomprehensible/incomplete than usual, and it is probably a better idea to look at the slides directly. In general, I usually use square brackets for comments.

Info You can find these notes on `http://poisson.phc.dm.unipi.it/~mennuni/Poland17.pdf` (but they could be moved; in case, check my Leeds webpage[1]). You can contact me at `mmrm@leeds.ac.uk`. This version has been compiled on July 10, 2017. To get the source code click on the paper clip.

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[1] Which does not exist yet, otherwise I would have linked that.

Chapter 1

Friday 30 June

1.1 Nadja Hempel – Mekler constructions in NIP and n-dependent theories

[joint with A. Chernikov]

1.1.1 Motivation

Examples of stable algebraic structures: ACF, SCF, DCF, free group, vector spaces.

Examples of NIP algebraic structures: RCF, ordered abelian groups, ACVF, $\mathbb{Q}_p$.

Definition 1.1. A formula[1] $\varphi(x; y_1, \ldots, y_k)$ has the *k-independence property* (or IP_k) iff there are parameters $(a_i^1)_{i\in\omega}, \ldots, (a_i^k)_{i\in\omega}$ and $(b_I)_{I\subseteq\omega^k}$ such that

$$\vDash \varphi(b_I, a_{i_1}^1, \ldots, a_{i_k}^k) \iff (i_1, \ldots, i_k) \in I$$

A theory is *k-dependent* iff no formula has the k-independence property.

Remark 1.2. NIP is the same as 1-dependent.

Remark 1.3. The k-dependent theories form an hierarchy: if $k \leq \ell$ then k-dependent implies ℓ-dependent.

Fact 1.4. The hierarchy is strict. For instance:

- The random graph is 2-dependent, but has IP.
- In general, the random k-hypergraph is k-dependent, but has IP_n for all $n < k$.

Question 1.5. Can one find algebraic examples, say of a 2-dependent theory with IP?

[1] Everything can be a tuple.

Yes:

Example 1.6 (Hempel 2015)**.** Fix a prime p and consider $G = \oplus_\omega \mathbb{F}_p$. Consider the structure $(G, \mathbb{F}_p, +, 0, \cdot)$, where $\cdot$ is a bilinear form defined as

$$(a_i)_{i\in\omega} \cdot (b_i)_{i\in\omega} \coloneqq \sum_{i\in\omega} a_i \cdot b_i$$

Proposition 1.7. The structure above has IP, is simple, and is 2-dependent.

Until now there were no algebraic examples witnessing strictness of the hierarchy.

Conjecture 1.8. There are no (super)simple strictly k-dependent fields for $k > 1$.

Theorem 1.9 (Hempel, 2015)**.** Let K be an infinite field.

1. If $\mathrm{Th}(K)$ is k-dependent, then K is Artin-Schreier closed
2. If K is PAC and not separably closed, then it has IP_k for all k.

The goal is to find groups that witness strictness of the hierarchy.

1.1.2 Mekler constructions

The basic idea is taking a graph with certain properties and "turn them into" 2-nilpotent groups of exponenent $p > 2$, p prime.

Start with a graph $\Gamma = (V, E)$. Then define a group $G(\Gamma)$ like this the set of vertices V of Γ are a set of generators, with the following relations relations:

- $[\forall a, b, c \in V\ [a, b], c] = 1$
- if $E(a, b)$, then $[a, b] = 1$
- $a^p = 1$

This is a general construction. Mekler constructions are done starting with a *nice graph*.

Definition 1.10. Γ is *nice* iff:

- it is triangle-free,
- it is square-free,
- for all a, b there is c such that if $a \neq b$ then there is $c \notin \{a, b\}$ such that $E(a, c)$ and $\neg E(b, c)$

Fact 1.11. The graph is interpretable in the group.

What about the other direction? There are at least some "tameness is preserved" results.

Theorem 1.12. Let Γ be a nice graph.

1. [Mekler 1981] Γ is λ-stable if and only if $G(\Gamma)$ is λ-stable.
2. [Baudisch/Pentzel 2002] Γ is simple if and only if $G(\Gamma)$ is.

To understand[2] $G(\Gamma)$, we need to introduce some equivalence relations.

Definition 1.13. Define

1. $g \sim h$ iff $C_G(g) = C_G(h)$ (i.e. if they have the same centralisers)
2. $g \approx h$ iff $g = h^i \cdot c$, for $1 \leq i < p$, $c \in Z(G)$.
3. $g \equiv_Z h$ if $g \cdot Z(G) = h \cdot Z$.

From now on, let $Z \coloneqq Z(G)$.

Remark 1.14. Every equivalence relation implies the previous ones in the list. Moreover, they are all $\emptyset$-definable.

Definition 1.15. An element g is

- *of type* q iff q is the number of $\approx$-classes in the $\sim$-class of g.
- *isolated* iff, for all h, if $g \approx h$ then $gZ = hZ$.

Theorem 1.16 (Mekler)**.** The following facts hold:

a) Every non-central element of $G(\Gamma)$ is either of type
   - 1^ν, i.e. type 1, non-isolated,
   - 1^ι, i.e. type 1, isolated,
   - $p - 1$ (automatically non-isolated), or
   - p (automatically non-isolated).

b) An element of $G(\Gamma) \setminus Z$ is $\approx$-equivalent to a vertex iff it is of type 1^ν.

c) For every element g of type p, there is an element b of type 1^ν which commutes with g. This is unique up to $\sim$-equivalence. We call such an element the *handle* of g.

Fact 1.17. A group elementarily equivalent to a $G(\Gamma)$ does not necessarily arise as a $G(\tilde{\Gamma})$.

[2] From now on denoted G up to further notice.

Anyway, it can be somehow controlled using the machinery above. What follows is slightly imprecise.

Let[3] $G \vDash \mathrm{Th}(G(\Gamma))$. This looks like a $G(\Gamma)$, except you have to

- add additional elements in the center,
- add elements of type p with some handle b,
- add elements of type 1^ι.

The goal now is writing $G = \langle X \rangle \times H$, where $J \leq Z$ is an elementary abelian group, and $X = X^{1^\nu} \cup X^p \cup X^{1^\iota}$.

Start with X^{1^ν} (called the 1^ν-transversal) set of representatives of each $\sim$-class of elements of type 1^ν.

Definition 1.18. An element in G is *proper* if it is no ta product of elements of type 1^ν.

Then let the p-transversal X^p be a set of representatives of $\sim$-classes of proper elements of type p which is maximal with the property that if $Y \in \mathscr{P}_{\mathrm{fin}} X^p$ is such that all elements of Y have the same handle, then Y is independent[4] modulo the subgroups generated by the elements of type 1^ν and the center.

The 1^ι-transversal X^{1^ι} is similarly defined.

$X = X^{1^\nu} \cup X^p \cup X^{1^\iota}$ is called a *transversal*, and any $Y \subseteq X$ is called a *partial transversal*.

Theorem 1.19. $G = \langle X \rangle \times H$, where $H \leq Z(G)$.

Fact 1.20. The commutator group G' equals $\langle X \rangle$.

One important ingredient of the proof is the following:

Lemma 1.21. In a model G of $\mathrm{Th}(G(\Gamma))$, let h be a bijection between two partial transversals Y and $h(Y)$ such that[5]

- $\mathrm{tp}_\Gamma(Y_{1^\nu}^{\approx}) = \mathrm{tp}_\Gamma(h(Y_{1^\nu})^{\approx})$, and such that
- h respects the 1^ν, p, 1^ι partes and the handles.

Then $\mathrm{tp}(Y) = \mathrm{tp}(h(Y))$.

This is what you use to transfer model-theoretic statements between the graph and the group (you can use the thing above to get the vertices, and the edges are there when they commute).

[3] Drop the convention $G = G(\Gamma)$.

[4] Z is an elementary abelian group, and so is $G(\Gamma)/Z$. This means basically that they are $\mathbb{F}_p$-vector spaces. So here "independent" means "linearly independent".

[5] The symbol $^{\approx}$ denotes the $\approx$-equivalence class.

Theorem 1.22 (Chernikov, Hempel, 2017)**.** Let Γ be a nice graph.

a) $\mathrm{Th}(\Gamma)$ is k-dependent iff $\mathrm{Th}(G(\Gamma))$ is k-dependent.

b) $\mathrm{Th}(\Gamma)$ is NTP_2 iff $\mathrm{Th}(G(\Gamma))$ is NTP_2.

Remark 1.23. Another result of Mekler says that for any relational structure [in a finite vocabulary?] there is a nice graph biinterpretable with it. Therefore, these are algebraic examples witnessing strictness of the hierarchy.

1.2 Franziska Jahnke – Dreaming about NIP fields

[joint with S. Anscombe]

1.2.1 Introduction

We are going to talk of a generalisation of the following

Conjecture 1.24 (Stable fields conjecture)**.** Every infinite stable field is separably closed.

(also, there is a conjecture that every infinite [supersimple?] field is PAC)

Conjecture 1.25 (Shelah)**.** Any infinite NIP field is real closed, separably closed, or "like the p-adic numbers".

There are two standard ways of making the conjecture above more formal. They differn by a single word.

Conjecture 1.26 (Conjecture A)**.** Let k be an infinite NIP field[6] neither separably closed nor reals closed. Then k admits a non-trivial henselian valuation.

Conjecture 1.27 (Conjecture A')**.** Let k be an infinite NIP field neither separably closed nor reals closed. Then k admits a non-trivial *definable*[7] henselian valuation.

In this talk we will see that these two conjectures are actually equivalent.

Remark 1.28. Conjecture A' implies Conjecture A, which implies the stable fields conjecture[8].

Fact 1.29 (Johnson)**.** Both conjectures A and A' hold iff "NIP" is replaced by "dp-minimal".

[6] In the sense that its L_{ring}-theory is NIP, but there may be extra structure.
[7] In the sense that the valuation ring $\mathcal{O}$ is L_{ring}-definable
[8] Which in particular seems to put them out of reach

1.2.2 Some facts about henselian fields and definable valuations

Fix a non-separably closed field $K \neq K^{\mathrm{sep}}$. Then all (non-trivial) henselian valuations on K induce the same topology.

Fact 1.30 (Endler/Engler)**.** There is a canonical[9] henselian valuation v_K on K, which is non-trivial if and only if K admits a non-trivial henselian valuation. If K admits an henselian valuation with separably closed residue field, then v_k is the coarsest such[10] and v_K is the finest henselian valuation on K. In this case, the residue field k_{V_K} is non-henselian[11], i.e. it does not admit any non-trivial henselian valuation.

Theorem 1.31 (Jahnke, Koenigsmann)**.** Let (K, v) be a non-trivial henselian valued field, with $K \not\models \mathsf{SCF}$ and $K \not\models \mathsf{RCF}$. Then there is some non-trivial L_{ring}-definable valuation ring O_w on K with $O_w \subseteq O_v$ or $O_v \subseteq O_w$.

More specifically:

Theorem 1.32. If $v = v_K$ is the canonical henselian valuation and k_{v_K} is separably closed or real closed, then $\mathcal{O}_{v_k} \subseteq \mathcal{O}_w$ (in particular, w is automatically henselian).

If $v = v_K$ and k_{v_K} is neither separably closed nor real closed, then $\mathcal{O}_w \subseteq \mathcal{O}_{v_K}$.

Note that the valuations there need not be henselian.

Definition 1.33. Let $M \prec N$ be L structures and $\varphi(x, y)$ an L-formula. An *externally definable* subset of M is a set of the form $\varphi(M^{|x|}, b)$, for $b \in N$. Define M^{Sh} as the expansion of M by predicates for all externally definable sets.

Example 1.34.

- If Γ is an ordered abelian group and $\Delta \leq \Gamma$ is a convex subgroup, then Δ is externally definable in Γ in $L_{\mathrm{oag}} = \{+, 0, \leq\}$ (just pick a saturated model realising the relevant types).
- If (K, w) is a valued field and v is a valuation on K with $\mathcal{O}_v \supseteq \mathcal{O}_w$, then $\mathcal{O}_v$ is definable in $(K, w)^{\mathrm{Sh}}$.

Fact 1.35 (Shelah)**.** If M is NIP, then so is M^{Sh}.

Corollary 1.36. If K is NIP (in L_{ring}), and (K, v_k) is henselian, then[12] k_{v_K} is NIP in L_{ring}.

Use Theorem 1.32? □

[9]In a valuation-theoretic point of view [I missed the exact sense]. It is not really functorial.

[10]I.e. it has the biggest valuation ring with the property that the residue field is separably closed.

[11]Because henselian valuations compose.

[12]Where v_K is the canonical henselian valuation.

1.2.3 A closer look at the conjectures

Proposition 1.37. Assume Conjecture A. Let K be an NIP field. Then, the canonical henselian valuation has residue field k_{v_K} either finite, real closed or separably closed.

Proof. Assume K is nip, infinite, $K \not\models \mathsf{SCF}$ and $K \not\models \mathsf{RCF}$. This can be done because if $K \models \mathsf{SCF}$ or K is finite, then v_K is trivial and so the thesis trivially holds, while if $K \models \mathsf{RCF}$ then so does k_{v_K}. By Conjecture A, K admits a non-trivial henselian valuation, so v_K is non-trivial. By Fact 1.30, k_{v_K} is either separably closed or non-henselian. By Corollary 1.36, k_{v_K} is NIP. Using again Conjecture A, it must be finite, separably closed or real closed. □

Corollary 1.38. Conjecture A implies Conjecture A'.

Proof. Assume K infinite, NIP, $K \not\models \mathsf{SCF}, \mathsf{RCF}$. Assume Conjecture A. By the Proposition, k_{v_K} is one of the following:

1. separably closed,
2. real closed, or
3. finite.

We now argue that in each of the three cases there is a definable henselian valuation.

For the first two cases, the w from the first part of Theorem 1.32 is a non-trivial henselian valuation.

In the case where k_{v_K} is finite, then v_K has no proper refinements, because on a finite field there is only the trivial valuation. It follows that $v_K = w$, for w as in the [second part of] Theorem[1.32]. □

1.2.4 What are the conjectural NIP fields?

Easy case: assume K is NIP, K is perfect, and the Galois group G_K is *small*[13], in the sense that K has only finitely many Galois extensions of degree n for every $n \in \ltimes$.

Johnson, as part of his classification of dp-minimal fields, showed:

Theorem 1.39 (Johnson)**.** Let (K, v) be henselian and defectless[14]. Suppose that

- $k_v \models \mathsf{ACF}$, or $k_v \models \mathsf{RCF}$, or $k_v \equiv L$ for some finite extension $L \supseteq \mathbb{Q}_p$,
- for all n, $|vK/n \cdot vK| < \infty$, and

[13] Also known as [*bounded*].

[14] I.e. for all algebraic $(L, w) \supseteq (K, v)$ we have $[K : L] = (wL : vK) : [k_w : k_v]$.

- if $\mathrm{char}(k_v) = p$, then the interval $[-v(p), v(p)]$ is p-divisible (in particular, in characteristic (p, p) the whole value group needs to be divisible, since $p = 0$ will have valuation ∞).

Then (K, v) is dp-minimal as a valued field.

(actually he also show that on an dp-minimal field there is a valuation satisfying those conditions, so it's an if and only if [kind of, did not understand properly])

Remark 1.40. In particular, this implies that every dp-minimal field is perfect (but this was already proven by Shelah) and has small absolute Galois group (in the sense of "G_K small" mentioned above).

Using Johnson's Theorem we have the following. Assume K NIP, perfect, G_K small, henselian wrt $v = v_K$. Since G_K is small, for all n prime with $\mathrm{char}(K)$ we have $|v_K K / n v_K K| < \infty$. Moreover, by a result of Jahnke, (K, v_K) is NIP (similar argument to what presented today). By a theorem of Jahnke and Simon, v_K will be defectless and the interval $[-v(p), v(p)]$ is either p-divisible (i.e. $\subseteq p \cdot vK$) or finite. In particular, $|\Gamma/n\Gamma| < \infty$ also for $p \mid n$.

If we now assume Conjecture A, we get that k_{v_K} is finite, real closed or separably closed. If k_{v_K} is finite, by KSW $\mathrm{char}(K) = 0$, and by a result of Simon then $K \equiv L((\Gamma))$, where $L \supseteq \mathbb{Q}_p$ is a finite extension and Γ is an ordered abelian group with $\forall n\ |\Gamma/n\Gamma| < \infty$.

Thus, Conjecture A implies that (K is NIP, perfect and with G_K small) if and only if K is dp-minimal.

1.3 Derya Çıray – Mild Parametrization in O-Minimal Structures

Suppose we have $X \subseteq \mathbb{R}^n$ and we want to count hay many rational points it contains of bounded height, where the height $H(q)$ of $q = a/b$ with $(a, b) = 1$ is $\max a, b$. So we want to bound the cardinality of

$$X(\mathbb{Q}, H) = \left\{ x = \left(\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n} \right) \in X \cap \mathbb{Q}^n, |a_i|, |b_i| \leq H \right\}$$

Role of the parametrisation in answering: $\varphi(0, 1) \to B([?])$ sends $t \mapsto (f(t), g(t))$ and X is the image of φ. Assume, at least for now, that φ is C^∞.

Let M be the set of monomials in 2 variables of degree $\leq d$. Let $\#M = d$, $m_j \in M$, $m_j = x^\alpha y^\beta$. Let $m_j(f(x_i), g(x_i)) = f(x_i)^\alpha \cdot g(x_i)^\beta$.

Let $I \subseteq (0, 1)$ and assume $\varphi(I)$ is not contained in any algebraic curves of degree $\leq d$. Let $x_1, \ldots, x_D \in I$, assume $(f(x_i), g(x_i)) \in \varphi(I)(\mathbb{Q}, H)$, and let Δ be the matrix $(m_j(f(x_i), g(x_i)))_{ij}$.

One can compute that

$$\frac{1}{H^{dD}} \leq \det \Delta \leq |I|^{\frac{-2}{[\text{something}]}} \cdot \underbrace{\left| \frac{m_j^{(i)}(\xi_{ij})}{i!} \right|}_{C(f,g,d)}$$

where $C(f,g,d)$ is a constant depending on f, g, d.

We can assume $|I| \geq (\star)$ [the thing above?] because if $|I| < \star$ then $\varphi(T)(\mathbb{Q}, H)$ is contained in an algebraic curve.

We can cover $\varphi(I)$ with $1 + 1/\star$ algebraic curves of degree d.

There are two [approaches? things to do?]: one is bounding hte number of hypersurfaces that contain the set (use smooth parameterisations), and the other one is bounding hte number of zero sets (number theoretic part).

Theorem 1.41 (Pila-Wilkie)**.** If R is an o-minimal expansion of $\mathbb{R}$ and X is definable in R, then for all $\varepsilon > 0$ there is c such that for all $H > 1$ we have $\#(X^{\text{tr}}(\mathbb{Q}, H)) \leq c \cdot H^{\varepsilon}$. Here $^{\text{tr}}$ denotes removing all the semialgebraic positive dimensional subsets ("tr" stands for "transcendental").

Conjecture 1.42 (Wilkie's conjecture)**.** Let $R = \mathbb{R}_{\exp}$. Then there are c_1, c_2 such that $|X^{\text{tr}}(\mathbb{Q}, H)| < c_i (\log H)^{c_2}$.

Theorem 1.43 (Pila 2006)**.** If X is a Pfaffian curve then $\#(X(\mathbb{Q}, H)) \leq c_1 (\log H)^{c_2}$.

This used *mild parametrisations*, i.e. functions $f \in C^\infty((0,1), [-1,1])$ such that for all N and $x \in (0,1)$

$$\frac{|f^{(n)}(x)|}{n!} \leq B^n n^{cn}$$

Example 1.44. Polynomials, restricted analytic functions, $e^{-\frac{1}{x}}$ are mild. A non-example is $x^{\frac{1}{2}}$, but the graph of $x^{\frac{1}{2}}$ has the mild parametrisation $t \mapsto (t^2, t)$. Or x^α for α irrational, but the graph can be parametrised by $t \mapsto (e^{-\frac{1}{t\alpha}}, e^{-\frac{1}{t}})$.

Definition 1.45. $X \subseteq \mathbb{R}^n$ *has mild parametrisations* iff there exist mild functions $\varphi_1, \ldots, \varphi_k \colon (0,1)^{\dim X} \to \mathbb{R}^n$ such that $\bigcup \operatorname{Im} \varphi_i = X$.

A structure R *admits (definable) mild parametrisation (m.p.)* iff all definable sets have m.p. (by means of definable functions).

[my question: an example of set with no mild parametrisation? some extremely fast growing function?]

Question 1.46. Which o-minimal expansions of $\mathbb{R}$ have (definable) m.p.?

It is known that:

- All reducts of $\mathbb{R}_{\text{an}}$ have definable m.p. (Jones, Miller, Thomas)
- There exists an o-minimal structure which does not admit definable m.p. (Thomas)
- Any polynomially bounded o-minimal structure that field of exponents $\neq \mathbb{Q}$ does not admit definable m.p. (Çıray)

The reason is x^α, with α irrational. The parametrisation above is not allowed in a polynomially bounded structure (of course this is not the proof). Assume φ, ψ are two mild functions definable in S (o-minimal, polynomially bounded) and $t \mapsto (\varphi(t), \psi(t))$ is a parametrisation of the graph of x^α. Then there are a, b in the fields of exponents of S such that

$$\frac{\varphi(t)}{x^\alpha} \to 1 \qquad \frac{\psi(t)}{x^\beta} \to 1$$

It follows that

$$\left(\frac{\varphi(x)}{x^a} \cdot \frac{1}{x^{b/\alpha} - a}\right)^\alpha \to 1$$

therefore $b/\alpha - a = 0$. Let's say $a \in \mathbb{R} \setminus \mathbb{Q}$. There is n_0 such that for all $n > n_0$ we have $\lim(x^a)^{(n)} = \infty$.

So $\lim \frac{\varphi^{(n)}x}{(x^\alpha)^{(n)}} = 1$ and $\varphi^{(n)}(x) = \infty$, and there are no definable m.p.

Question 1.47 (Open)**.** Does $\langle \mathbb{R}, x^\alpha \rangle$ admit m.p.?

Theorem 1.48 (Rolin, Servi)**.** If a family A of real functions satisfies "various" properties, then $\mathbb{R}_A$ (restrictions to boxes of this family) is o-minimal, polynomially bounded and model-complete.

Theorem 1.49 (Rolin, Servi)**.** Every bounded definable set in $\mathbb{R}_A$ admits parametrisation by means of functions whose components are in A.

It follows that

Fact 1.50. If everything in A is mild, then $\mathbb{R}_A$ admits m.p.

Corollary 1.51. $\mathbb{R}_G$ admits m.p. ([Gevrey?] functions).

Denjoy-Corleman classes. Let M be an increasing sequence in $\mathbb{R}^{>0}$, with $m_0 = 1$ and m_{n+1}/m_n increasing.

Let $C_i(M)$ be the set of $f\colon [-i, i] \to \mathbb{R}$ such that there are $U \supseteq [-i, i]$, $g\colon U \to \mathbb{R}$, suc hat $f \restriction [-i, i] = f$ and there is $A > 0$ such that $|g^{(n)}(x)|/n! \leq A^n m_n$.

Fact 1.52. Let $C(M) = \bigcup_i C_i(M)$. If $C(M)$ is a quasianalytic class (i.e. the Taylor map is injective). If in addition $\sum m_n/(m_{n+1}(n+1)) = \infty$ (call this a q.a.d.c. class) then $\mathbb{R}_{C(M)}$ is o-minimal.

Fact 1.53. There is a q.a.d.c. class which is not mild.

1.4 Grzegorz Jagiella – Definable topological dynamics and variants of definable amenability

Let G a group acting by homeomorphisms on a compact Hausdorff topological space X. Suppose there is a dense G-orbit (G-ambit). Call Y a subflow if [...]. We are interested in minimal subflows. They exist because [...] [Definition of Ellis enveloping semigroup $E(X)$ and basic properties].

Recall that $I \subseteq E(X)$ is a minimal subflow if and only if it is a minimal left ideal, and that these guys decompose in the disjoint union $I = \bigsqcup u * I$ where u ranges among the idempotents, and all the $u \cdot I$ are isomorphic. Call them *ideal groups*, and call their equivalence class the *Ellis group* of (G, X).

Now let T an arbitrary L-theory, $M \vDash T$ and G an M-definable group. Consider the action $G(M) \curvearrowright S_G(M)$. The problem is that its enveloping semigroup is not easily understood.

We instead study another flow: assume that T is NIP and pass to M^{ext}.

Theorem 1.54 (Shelah)**.** If T is NIP, then T^{ext} has q.e. and types over M^{ext} are definable.

Now consider $G(M) \curvearrowright S_G(M^{\text{ext}})$. Because of definability of types, we have

Theorem 1.55 (Newelski)**.** In this case $E(S_G(M^{\text{ext}})) \approx S_G(M^{\text{ext}})$.

Moreover, for $p, q \in S_G(M^{\text{ext}})$ we have that $p * q$ is the type of a pair (a, b) where $a \vDash p$ and $b \vDash q \mid Ma$, i.e. $(a, b) \vDash p \otimes q$ [WARNING, he is writing it "backwards"].

Conjecture 1.56 (Newelski)**.** Under "nice" assumptions, the Ellis group of $S_G(M)^{\text{ext}}$ is isomorphic to G/G^{00}.

"Nice" here assumes NIP (so G^{00} makes sense), and it implies that G^{00} does not change.

Remark 1.57. In later investigations, it turned out that maybe it makes more sense to look at G^{∞} instead.

There is a natural candidate, the map sending p to p/G^{00}.

Remark 1.58. G/G^{00} does not depend on M, so when the conjecture holds, the Ellis group is an invariant of the theory and of the group.

Example 1.59. The Conjecture fails in

1. $\mathrm{SL}(2, \mathbb{R})$ (Gismatullin, Penazzi, Pillay) (the Ellis group has two elements, but G/G^{00} is trivial)

2. $\mathrm{SL}(n, -)$, $\mathrm{GL}(n, -)$, and in general groups admitting a general decomposition (Jagiella)

Let's consider another conjecture

Conjecture 1.60. Let $N \succ M$ and assume that the Ellis groups of $S_G(M^{\text{ext}})$ and $S_G(N^{\text{ext}})$ are isomorphic.

[my question: under some natural map?] Problem: there is no canonical way to pass from types over M^{ext} to types over N^{ext}; even the language is different. So a priori they are different kinds of objects.

But one can do this construction. If $M^{\text{ext}} \prec N$ (in $L_{M,\text{ext}}$, consider then N^{ext} in $L_{N,\text{ext}}$. Now we can extend types in $S_G(M^{\text{ext}})$ to types in $S_G(N)$ and then to $S_G(N^{\text{ext}})$. So now we have a natural restriction map $r \colon S_G(N^{\text{ext}}) \to S_G(M^{\text{ext}})$.

Question 1.61. *

1. Is r an homomorphisms of groups (when restricted to ideal subgroups)? Maybe under additional assumptions?

2. Does r take ideal subgroups in ideal subgroups? Does the preimage under r do that?

This turns out to be the case under some hypotheses.

Definition 1.62. G is *definably amenable* iff there is a global probabilistic finitely additive measure on the definable[15] subsets of G which is G-invariant.

Theorem 1.63 (Chernikov, Simon)**.** If T is NIP and G is definably amenable, then it satisfies the first conjecture, and in fact the map $p \mapsto p/G^{00}$ is an isomorphism.

For purposes that will be clear later, we are interested in these two special cases of definable amenability.

Definition 1.64. G has *fsg* (finitely satisfiable generics) iff there is a global type p and a small model M such that all G-translates of p is finitely satisfied in M.

Example 1.65. Definably compact groups in o-minimal or p-minimal theories are fsg.

Recall that a subset of a group is *generic* iff finitely translates of it cover the group, and that a type is iff all of its formulas are.

Fact 1.66. Suppose G is fsg and M is any small model.

1. $\text{Gen}_G(M) \neq \emptyset$

2. $\text{Gen}_G(M^{\text{ext}}) \neq \emptyset$ (and in fact it consist of unique extensions of generics above)

[15]Since we said "global", we mean with parameters from the monster.

3. If $N \succ M$ then $\mathrm{Gen}_G(N)$ is precisely the set of coheirs of elements of $\mathrm{Gen}_G(M)$

4. $\mathrm{Gen}_G(M^{\mathrm{ext}})$ is a two-sided ideal in $S_G(M^{\mathrm{ext}})$.

Remark 1.67. G is fsg iff there is a [unique generic measure satisfying the definition of definable amenability?]

Theorem 1.68 (Newelski, no assumptions on T)**.** If $\mathrm{Gen}_G(M^{\mathrm{ext}}) \neq \emptyset$, then it is the unique minimal subflow of $S_G(M^{\mathrm{ext}})$ and the first Conjecture holds with the natural map as the isomorphism.

[Pillay remarks that without NIP there are more technical things; it should be the fact that you need to work with quantifier free ext-types; I think you also need to consider $G^{00}(M)$]

Definition 1.69. G is *definably extremely amenable* iff it is definably amenable and this is witnessed by a $\{0,1\}$-measure, i.e. a G-invariant type.

Fact 1.70.

1. If $p \in S_G(\mathfrak{U})$ is G-invariant, then $p \restriction M$ is $G(M)$-invariant.

2. If $p \in S_G(M)$ is G-invariant, then there is $p \subseteq p' \in S_G(M^{\mathrm{ext}})$ which is $G(M)$-invariant.

Fact 1.71. If $p \in S_G(M)$ is $G(M)$-invariant *and definable*, then for any $N \succ M$ we have that the unique heir $p \mid N$ is $G(N)$-invariant.

Now consider the following problem. Suppose $G = F \cdot H$ (everything definable) where $F \cap H = \{e\}$. We want to describe the topological dynamics of G in terms of the ones of F and H, and in terms of how the two interacts with each other. We have actions $G \curvearrowright G/H$ and $H \curvearrowright G/H$ (consider the restriction of the action on G). Now $G/H \sim F$, so we can understand this as an action $H \curvearrowright F$. In a similar fashion, we can consider an action $*_1 \colon S_H(M^{\mathrm{ext}}) \curvearrowright S_F(M^{\mathrm{ext}})$.

1. Let [t?] action of $H(M)$ on types in $S_H(M^{\mathrm{ext}})$ and their heirs.

2. Let [t?] an two-sided action of $F(M)$ on types on $S_F(M^{\mathrm{ext}})$ and their coheris.

In the above with "types" we actually mean "almost periodic types", i.e. types belonging to some minimal subflow.

What happens if F is fsg and H is definably extremely amenable? (G is not automatically definably amenable)

Theorem 1.72. Suppose $G = F \cdot H$, $F \cap H = \{e\}$, F is fsg, H is definable extremely amenable, $p \in S_H(M^{\mathrm{ext}})$ is $H(M)$-invariant.

1. There is a unique $I \subseteq S_F(M^{\text{ext}})$ such that $I * p$ is a minimal subflow of $S_G(M^{\text{ext}})$.

2. $I \supseteq \text{Gen}_F(M^{\text{ext}})$. Actually $I = \text{cl}(p *_1 \text{Gen})$.

3. $\text{Gen}_F(M^{\text{ext}}) * p$ decomposes into a disjoint union of [ideal?[also, in what sense "ideal"?]] subgroups.

4. An ideal subgroup is (in a certain natural way) isomorphic[16] to $p *_1 \text{Gen}(M^{\text{ext}})/F^{00} \cdot X/F^{00}$, where X depends on the ideal group.

As an application, assume that G is definable in an o-minimal expansion of an RCF.

1. Definably compact groups are fsg (Newelski)

2. Torsion-free groups are definably extremely amenable (Conversano, Pillay)

3. (GPP counterexample)

4. Compact/torsion-free decomposition $G = K \cdot H$ definable over $\mathbb{R}$. Then the Ellis group is isomorphic to $N_G(H) \cap K)/K^{00}$ (Jagiella[?])

5. Same set up over $R \succ \mathbb{R}$ (Yao)

6. Definably amenable groups (Pillay-Yao) (then generalised by Chernikov and Simon with no assumption of o-minimality)

There are similar applications in $\mathbb{Q}_p$ and p-minimal extensions.

[16]The isomorphism is of groups, need not respect the topology.

Chapter 2

Saturday 1 July

2.1 Junguk Lee – Survey on the first homology groups of strong types in model theory

[slides] Assume $T = T^{\mathrm{eq}}$, $\emptyset = \operatorname{acl}(\emptyset)$. Fix a onster, a strong type p over $\emptyset$, and a ternary $\mathop{\smile\!\!\!\!\!\!|}^*$ satisfying finite character, normality, symmetry, transitivity, extension.

Consider the category $\mathcal{C}$ with objects the small subsets of $\mathfrak{U}$ and morphisms the elementary maps.

For a finite set s of naturals, consider the power set as a category by saying that the morphisms are the inclusions.

Consider a functor $\mathscr{P}(s) \to \mathcal{C}$. Write f_v^u for f of the inclusion $u \subseteq v$, and $f_v^u(u)$ for $f_v^u(f(u))$.

Definition 2.1. A $*$-independent n-simplex in p is a functor $f\colon \mathscr{P}(s) \to \mathcal{C}$ such that

- $|s| = n + 1$
- for $i \in s$, $f(\{i\}) = \operatorname{acl} a$ for some $a \vDash p$
- For $\emptyset \neq u \subseteq s$, $f(u) = \operatorname{acl} \bigcup_{i \in u} f_u^{\{i\}}(\{i\})$ and [...]
- [...]

[too fast; see slides]
[defines chain complexes and takes homology in the usual way]
[computes the first homology group; it involves the notion of *shell*]
[the natural action of $\operatorname{Aut}(\mathfrak{U})$ on the first homology group is trivial]

Fact 2.2. The first homology group of a strong type is a quotient of $\operatorname{Aut}(\mathfrak{U})$, by [missed by what; anyway there is some topology going on and the kernel should be closed if the theory is G-compact.]

[my question: does $H_1^*(p)$ depend on $\mathfrak{U}$?]

Also, it is a quotient of $\mathrm{Gal}_L(T)$.

An application gives conditions for a strong type to be a Lascar strong type.

[Jan later told me the point of this homology is measuring how far you are to have amalgamation]

2.2 Charlotte Kestner – An introduction to non-forking formulas in NIP theories

[Joint with G. Boxall] This is an introductory talk, and is intended as preliminary to Gareth's talk.

2.2.1 NIP Theories

Definition 2.3. Let $\varphi(x, y)$ be a formula. We say that a set of $|x|$-tuples is *shattered* by φ iff there is $(b_I \mid I \subseteq A)$ such that $\vDash \varphi(a; b_I)$ if and only if $a \in I$.

A formula has NIP if it does not shatter any infinite set. A theory has NIP if every formula has NIP. The VC-dimension of $\varphi(x; y)$ is the dimension of the largest set it can shatter.

Example 2.4. DLO has NIP. For instance, if $\varphi(x; y_1, y_2)$ is $y_1 < x < y_2$, then VC-dim$(\varphi(x; y)) = 2$ [picture and explanation]

Theorem 2.5 ((p, q)-theorem)**.** Given integers $p \geq q$ there is $n \in \mathbb{Z}$ such that the following happens. Suppose $\varphi(x; y)$ has dual VC-dimension[1] $< q$ and let $W = \{b \in M^{|y|} \mid \vDash \exists x\ \varphi(x; b)\}$. Suppose that for Y finite subset of W, for $Y_0 \subseteq Y$ such that $|Y_0| = p$ there is $Y_1 \subseteq Y_0$ such that $|Y_1| = q$ and $a \in M^{|x|}$ such that $Y_1 \subseteq \varphi(a, M)$. Then there are $a_0, \ldots, a_{n-1}$ such that $Y \subseteq \bigcup_{i<n} \varphi(a_i, M)$.

2.2.2 Distality

Definition 2.6. A theory T is *distal* iff whenever[2] $I+b+J$ is $\emptyset$-indiscernible, with I, J infinite, then for any A such that $I + J$ is A-indiscernibl, then $I + b + J$ is A-indiscernible.

[my question: does this definition imply NIP?]

Example 2.7. DLO is distal. Suppose I and J are made of singletons. If $I + J$ is A-indiscernible it means there is no element of A in between them. So this still holds for $I + b + J$.

[1] I.e. the VC-dimension of $\varphi(y; x)$.

[2] Here b is a single element.

Theorem 2.8 (Chernikov, Simon)**.** Suppose T is NIP. The following are equivalent:

1. T is distal.
2. T has strong honest definitions, i.e. for any $\varphi(x, y)$ there is $\theta(x, z)$ with $|z| = k|y|$ such that for any a, c and finite $B \subseteq C$ with $|B| \geq 2$ there is $c \in C^k$ such that $\models \theta(a, c)$ and $\theta(x, c) \vdash \text{tp}_{\varphi}(a/B)$.

2.2.3 Non-forking formulas

[definition of forking]

Theorem 2.9 (Shelah, Chernikov, Kaplan)**.** Let T be NIP.

- Every M-non-forking formula extends to an M-invariant[3] global type.
- Existence of strict Morley sequences, i.e. the following. Given $b \in \mathfrak{U}$ and $M \models T$ small there is an indiscernible sequence $b = b_0, b_1, \ldots$ which is strictly non-forking over M.

A consequence of strict non-forking[4] that will be used is that if $\varphi(x, b)$ forks over M, then it is witnessed by the sequence above, i.e. $\{\varphi(x; b_i) \mid i \in \omega\}$ is inconsistent.

Theorem 2.10 (Chernikov, Kaplan)**.** In NIP theories, forking equals dividing over models.

2.2.4 The definable (p, q)-conjecture

Conjecture 2.11 ("Definable (p, q)-theorem")**.** Let $M \prec^{+} N$ be NIP L-structures, N saturated over M. Suppose $\varphi(x, b)$ does not fork over M. Then there is an L_M-formula $\psi(y) \in \text{tp}(b/M)$ such that $\{\varphi(x, b') \mid\models \psi(b')\}$ is consistent.

Remark 2.12. If instead of ψ you plug the whole $\text{tp}(b/M)$ then this is known to be true.

Why is that called "definable (p, q)-theorem"?

Fact 2.13. If T is NIP, the (p, q)-theorem implies the following. If $\varphi(x, b)$ is non-forking, there are W a definable set and $\{W_i \subseteq W\}_{i=1}^{n}$, not necessarily definable, such that $W = \bigcup_{i=1}^{n} W_i$ and for all $i \leq n$ the set $\{\varphi(x, b') \mid b' \in W_i\}$ is consistent.

Fact 2.14. The definable (p, q)-conjecture is true in stable theories.

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[3] Over a small model.

[4] We are not going to see the definition.

Proof. If $\varphi(x,b)$ does not fork over M, it extends to a global M-invariant type $p(x)$, which by stability is M-definable. Use the defining scheme of φ as ψ. Then $\{\varphi(x,b') \mid \models d_p\varphi(b')\}$ is a subset of p, hence consistent. $\square$

Theorem 2.15 (Boxall, Kestner)**.** The definable (p,q)-theorem holds in distal theories.

The proof uses strong honest definitions and strict Morley sequences. This is one of the ingredients:

Lemma 2.16. There are $\theta(x,z)$, $k \in \mathbb{N}$ such that $|z| = k|y|$ and a complete type $r(z)$ over M such that for each finite set $B \subseteq N^{|y|}$ of realisations of $\mathrm{tp}(b/M)$ and all $a \in N^{|x|}$ if for all $b \in B$ we have $N \models \varphi(a,b)$ there is c such that

1. $N \models r(c)$,
2. $\theta(x,c)$ does not fork over M,
3. $N \models \theta(a,c)$,
4. for all $b \in B$ we have $N \models \forall x\ \theta(x,c) \to \varphi(x,b)$.

Another ingredient is

Proposition 2.17 (Simon)**.** Let $M \prec^+ N$ be NIP. If $\varphi(x,b)$ does not divide over M, then there is $\psi(y) \in \mathrm{tp}(b/M)$ and a *finite* $A_\psi \subseteq N$ such that for each $b' \in \psi(M)$ there is some $a \in A_\psi$ such that $\varphi(a,b')$.

[proof sketch of the main theorem]

2.3 Mike Haskel – Stability in hyperimaginary sorts

2.3.1 Hyperimaginaries

Definition 2.18. An *hyperimaginary* sort is a type-definable set X modulo a type-definable equivalence relation.

The real core is the fact that the *equivalence relation* is type-definable, as opposed to definable, otherwise it would just be an imaginary.

Example 2.19. In RCF the relation of being infinitesimally close.

Definition 2.20. Let X/E be an hyperimaginary sort, and let $\Gamma(x)$ be a partial type in X. Say that Γ is a *partial type in* X/E if $\Gamma(x_1) \wedge x_1 E x_2 \to \Gamma(x_2)$. A *complete type in* X/E is a maximal consistent partial type.

Fact 2.21. Partial types form a topology of closed sets on the space of complete types, and the resulting space is compact Hausdorff.

Remark 2.22. It is not necessarily totally disconnected, so it is not a Stone space in general.

Some problems we may encounter is:

- No notion of "definable".
- The complement of a type-definable set need not be type-definable.
- As an instance of that, since equality is now only type-definable (it is a type-definable equivalence relation), inequality is not type-definable.

Some strategies one can use to deal with these problems are:

- Using real analysis and/or continuous logic: the point is that while Stone spaces can be understood in terms of Boolean algebras, to deal with compact Hausdorff spaces which are not Stone spaces we have to resort to real analysis.
- Replace properties of formulas with properties approximated by a family of formulas
- Replace "infinite" with "indiscernible". For instance if you have infinitely many things which are pairwise inequivalent, and say you want to use compactness to do something, you have the problem that inequivalence may be witnessed by different formulas. Of course this problem disappears if you use indiscernible things instead of merely infinite ones.

What we are going to see is somehow replicative of continuous logic, but the speaker thinks the approach here provides more intuition about the problem at hand.

2.3.2 Stability

Definition 2.23. An hyperimaginary sort X/E is *stable*[5] if there is no indiscernible witness to the order property where one of its sorts (i.e. the a's or the b's) is X/E.

In this context, since we do not have definable sets, "the order property" means you do not have two indiscernible sequences (a_i) and (b_i) such that the type of $a_1 b_2$ is different from the type of $a_2 b_1$.

Example 2.24. Consider the theory of $(\mathbb{R}, +)$ with predicates for intervals $(-1/n, 1/n)$. Let $I = \bigwedge_n(-1/n, 1/n)$. Consider R/I. This is

- NIP (inasmuch a reduct of RCF)

[5] A.k.a. stable, stably embedded.

- unstable (using the predicates you can define the positive numbers (the one at distance less than 1 from 1 [probably sligtly different, take 3. . .])
- R/I is stable (it somehow reduces to $(\mathbb{R}, +)$, which is known to be stable)

Now we want to do local stability; since we don't have negation (we would like to say that φ goes in one direction, $\neg\varphi$ in the other), we have to use pairs of formulas.

Definition 2.25. Let X/E be an hyperimaginary sort, and $(\varphi(x, y), \psi(x, y))$ a pair of formulas (y can be a tuple from any sort). Say the pair is *respectful* iff $\varphi(x_1, y) \wedge x_1 E x_2 \wedge \psi(x_2, y)$ is inconsistent.

Definition 2.26. A pair $(\varphi(x, y), \psi(x, y))$ is *stable* iff there is no infinite witness to the order property for (φ, ψ).

(we can say "infinite" instead of "indiscernible" because we are talking about pairs of formulas: instead of saying "φ in one direction, $\neg\varphi$ in the other", replace $\neg\varphi$ with ψ)

Theorem 2.27 (Haskel)**.** *X/E is stable if and only if every respectful pair $(\varphi(x, y), \psi(x, y))$ is stable as [?] varies.*

Proof. ($\Rightarrow$) We have $(\varphi(x, y), \psi(x, y))$ respectful, and by usual Ramsey arguments we may assume the sequence witnessing instability is indiscernible. Say $\models \varphi(b_2, a_3)$ and $\models \psi(a_1, b_2)$. Write $\exists x'\ xEx' \wedge \varphi(x', y)$ and use the definition of "respectful".

($\Leftarrow$) Assume that $b_0 a_1 b_2 a_3 \ldots$ witnesses instability of X/E, i.e $\mathrm{tp}(b_2 a_3)$ and $\mathrm{tp}(b_2 a_1)$ are distinct. Call those types p and q. By compactness there is $\varphi \in q$ such that $\neg\varphi \in p$. We have $p(x, y) \wedge x_1 E x_2 \to \neg\varphi(x_2, y)$. □

Definition 2.28. Let Γ be a collection of formulas (resp. bipartite formulas in $(x; y)$). Γ is *respectful* (resp. *stable*) iff for all $\varphi \in \Gamma$ there is $\psi \in \Gamma$ such that $(\psi, \neg\varphi)$ is respectful (resp. stable).

Remark 2.29. Some remarks:

- Γ is respectful if and only if it is a partial type on X/E.
- $\{\varphi\}$ is stable if and only if φ is a stable formula.

Now we want to develop heirs and coheirs. Suppose $p(x)$, $q(y)$ are complete types over some model M, and $\Gamma(x, y)$ is a partial type. We want to build $(p \otimes q)(x, y)$. We want to say whether $p \otimes q \models \Gamma$. The strategies are:

- Choose $b \models q$ and extend p to $\bar{p}$ over Mb, finitely satisfiable in M. Then ask if $\bar{p} \models \Gamma(x, b)$.

- Choose $a \vDash p$, and extend q to $\bar{q}$ over Ma, finitely satisfiable in M. Then ask if $\bar{p} \vDash \Gamma(a, y)$.

These two different strategies in general give different answers, and even worse than that we may have to make choices to defines the extension (e.g. $\bar{p}$). In other words coheirs need not be unique. These problems disappear if we use stable families:

Theorem 2.30 (Haskel)**.** If $\Gamma(x, y)$ is a stable family, the above strategies give the same answer for $p \otimes q \vDash \Gamma(x, y)$, and moreover there is no dependence on choice of extensions.

Proof. Assume $b \vDash q$ and $\bar{p} \vDash \Gamma(x, b)$ is M-finitely satisfiable. Assume $a \vDash p$ and $\bar{q} \vDash \neg\varphi(a, y)$ for some $\varphi \in \Gamma$. We get $\psi \in \Gamma$ with $(\psi, \neg\varphi)$ stable. Now use a classical stability theory argument to get a witness of the order property for $(\psi, \neg\varphi)$. □

Note that this says, all at once "unique heirs, unique coheirs, and they coincide".

Theorem 2.31 (Haskel)**.** Let $p(x)$ be a complete type over $B \subseteq Y$ and $\Gamma(x, y)$ a stable family. Then there is a $\Delta(y)$ type-definable over $\leq |\Gamma|$ parameters from B and for all $b \in B$ we have

$$p \vDash \Gamma(x, b) \iff \Delta(b)$$

There is also a stationarity result. Recall that in stable theories types over models M have unique M-invariant extensions. Here it is slightly different, M has to be $|\Gamma|^+$-saturated.

2.3.3 Motivation

Why did all this come up in the first place? An analogous of G^{00}, but instead of bounded quotients we talk of stable quotients.

Theorem 2.32 (Haskel, Pillay)**.** Let G be an NIP definable[6] group. Then there is a type-definable subgroup G^{st} which is smallest among those for which G/G^{st} is stable (even with parameters). Moreover, G^{st} is over φ and normal in G.

The point is that G/G^{00} gives an invariant which is a compact group, and we know a lot about compact groups. Here we would like to take advantage of the theory of stable groups, but this quotient is not really a stable group, it is a stable hyperimaginary group, and that is why we need the technology above.

[6] Maybe it is even true for type-definable.

2.4 Jan Dobrowolski – Dp-minimality

We will actually focus on dp-minimal groups. The main result is

Theorem 2.33 (Dobrowolski, (Goodrick?))**.** If $(G, \leq)$ is a inp-minimal (left-)ordered group then G is abelian.

This answers a question of G. Simon already proved the case of a bi-ordered group.

Definition 2.34. An inp-pattern in a partial type $\pi(x)$ of depth κ is a sequence of formulas $(\varphi_i(x, y_i))_{i<\kappa}$ together with an array of parameters $(a_{ij})_{i<\kappa, j<\omega}$ satisfying the following:

a) For each $i < \kappa$ there is $k_i < \omega$ such that $\{\varphi_i(x, a_{i,j}) \mid j < \omega\}$ is k_i-inconsistent [my question: with π?].

b) For every $\eta\colon \kappa \to \omega$ the set $\{\varphi_i(x, a_{i,\eta(i)}) \mid i < \kappa\}$ is consistent with $\pi(x)$.

Definition 2.35. inp-rk$(\pi(x)) \leq \kappa$ if there is no inp-pattern of depth $\kappa + 1$.

We mostly work with the case of rank 1.

Definition 2.36. inp-rk(T) := bdn(T) := inp-rk$(x = x)$ (in 1 variable). (bdn stands for "burden").

Fact 2.37.

- If T is NIP, then bdn(T) = dp-rk(T).
- T has IP if and only inf dp-rk $= \infty$.

The question was originally stated for dp-minimal, but we actually work with inp-minimal.

Fact 2.38. "dp-minimal" is the same as "inp-minimal and NIP".

Example 2.39. Some dp-minimal theories are:

- ACVF (in the valued field sort[7])
- $\mathbb{Q}_p$
- $(\mathbb{Z}, +, <)$
- $(\mathbb{R}, +, \cdot)$

[7] In multi-sorted context, one should actually say whether sorts are dp-minimal.

Fact 2.40. More generally, an ordered abelian group $(G, \leq)$ is dp-minimal if and only if for all prime p we have $[G : pG] < \omega$ ([someone; Jahnke maybe?], Simon, Walsberg). (without the order, if it happens for two distinct primes you get a pattern of depth 2)

Definition 2.41. T is *strong* iff there is no infinite depth pattern.

Fact 2.42 ((Haler?), Hasson)**.** An ordered abelian group is strong if and only if it is of finite inp/dp-rank (they are the same because those are always NIP), if and only if for almost all primes p the index $[G : pG]$ is finite, if and only if (a technical condition).

Fact 2.43 (Kaplan, Onshuus, Usvyatsov)**.** If dp-rk$(a_i/A) \leq \kappa_i$ for $i \in \{1, 2\}$, then dp-rk$(a_1a_2/A) \leq \kappa_1 + \kappa_2$.

Fact 2.44 (Levi, Kaplan, Simon)**.** If G is ω-categorical and dp-minimal then it is nilpotent-by-finite.

The thing above is open for general NIP (conjectured by Chernikov). For inp-minimal ω-categorical it is also true, by Wagner.

Fact 2.45 (Simon)**.** If G is an inp-minimal group and H, K are definable subgroups of G, then $[H : H \cap K]$ and $[K : H \cap K]$ are finite.

Proof sketch. If not, find $h_i \in H$, $k_i \in K$ with $h_i^{-1}h_j \notin K$, $k_ik_j^{-1} \notin H$ for $i \neq j$, and use as pattern $h_0K, h_1K, \ldots$ for the first row, $Hk_0, Hk_1, \ldots$ for the second row. This contradicts inp-minimality. □

Fact 2.46 (Simon)**.** If G is an inp-minimal group, then it is abelian by finite exponents witnessed by a definable group, i.e. there is an abelian, definable, normal subgroup N of G such that G/N has finite exponent.

Question 2.47 (Goodrick)**.** Is every dp-minimal (left-)ordered group abelian?

Fact 2.48 (Simon)**.** Yes for bi-ordered groups (i.e. the order is invariant on the left and on the right).

Lemma 2.49. If H is a definable subgroup of G, with G inp-minimal left-ordered, then the convex hull of H is a union of finitely many right cosets (fmrc) of H.

We do not see the proof, but it uses inp-minimality.

Remark 2.50. The convex hull $h(H)$ of H need not be a group in general.

We now look at the proof of the main Theorem.

It is enough to show that $[G : Z(G)] < \omega$; this is because if so, then $|G'| < \omega$ by general group theory. But since the group is ordered it is torsion free, so $G' = \{e\}$, i.e. G is abelian.

Let N be given by Fact 2.46. We claim that $[G : N] < \omega$. We will show that $G = h(N)$. Take $e < g \in G$. By left invariance we have

$$N \ni e < g < g^2 < \ldots < g^\ell \in N$$

So the convex hull is covered by finite many right cosets, hence the index is finite as promised (by dp-minimality, see above).

Every coset of N is cofinal in G, because N is cofinal in G.

Claim. For any $e < x_0 \in G$, the interval $[e, x_0]$ is covered by finitely many right cosets of $Z(G)$

Proof. Fix $e < x_0 \in G$. By the property above there are $x_0 < x_1, \ldots, x_n$ such that $\langle x_0, x_1, \ldots, x_n, N\rangle = G$. For every $i \leq n$, since $x_i \in C(x_i)$, we have that $x_0 \in h(C(x_i))$ (it is in the interval $[e, x_i]$. So we have ($[\ldots]$ is still intervals, not commutators)

$$[e, x_0] \subseteq \bigcap_{i\leq n} h(C(x_i))$$

But since $h(C(x_i))$ is covered by fmrc of $C(x_i)$. So fmrc of

$$N \cap \bigcap_{i\leq n} C(x_i)$$

covers $\bigcap_{i\leq n} h(C(x_i))$. Note that $N \cap \bigcap_{i\leq n} C(x_i) \subseteq Z(G)$, because $\langle x_0, x_1, \ldots, x_n, N\rangle = G$. □

To complete the proof, assume $[G : Z(G)]$ is infinite. Choose infinitely many positive representatives $e < b_0, b_1, \ldots$ in distinct cosets of $Z(G)$. Assuming the structure is ω_1-saturated, choose $x_0 > \{b_i \mid i < \omega\}$. Then $[e, x_0]$ cannot be covered by finitely any cosets of $Z(G)$ (it contains infinitely many representatives of pairwise distinct cosets).

Corollary 2.51. inp-minimal (pure) ordered groups are automatically dp-minimal.

Proof. Ordered abelian groups are NIP. □

Question 2.52. What about higher dimensions? Can one say something interesting, for instance, about finite dp-rank/inp rank (bi?)-ordered groups?

Already in dimension 2 we do not get abelianity.

Example 2.53. Let K_2 be the Klein bottle group, i.e. the one presented by $\langle x, y \mid yx = xy^{-1}\rangle$. This is isomorphic to $(Z \oplus Z, \odot)$, where

$$(n, m) \odot (a, b) = (n + a, b + (-1)^a \cdot m)$$

and the order is the lexicographical one. This is interpretable in $(\mathbb{Z}; +, <)$ (and those are known to be abelian-by finite; actually by 2) and has dp-rank 2. It is not abelian.

Proposition 2.54. If $\oplus bdn(G) \leq n \in \omega$ then there do not exist definable groups $D_0, \ldots, D_n$ such that if we put N_i to be the normal subgroup generated by D_i, then for each $i \leq n$ we have $[D_i/N_0 \ldots N_{i-1}N_{i+1} \ldots, N_n] \geq \omega$.

Example 2.55. This does not hold in the free group, say on generators $x_0, x_1, \ldots$, as witnessed by $D_i = \langle x_i \rangle$, which are definable as $C(x_i)$. This group is not strong (the generic type has infinite rank).

One would apply this to the ordered case. The speaker, so far, has not be able to, but he has obtained this by-product:

Proposition 2.56. If G is stable of finite burden (equivalently, finite weight), then there are finitely many abelian definable subgroups $A_0, \ldots, A_k$ such that[8] $G/\langle A_0, \ldots, A_k \rangle_N$ has finite exponent.

Question 2.57. Is the conclusion of the Proposition above true for some context when there is an order, e.g. rosy theories? O-minimal? Simple? Distal?

[8]That is "normal subgroup generated by". Note that it is not necessarily definable.

Chapter 3

Sunday 2 July

3.1 Slavko Moconja – Vaught conjecture for binary weakly o-minimal theories

Recall that

Conjecture 3.1 (Vaught, 1961)**.** If T is a complete theory with infinite models in a countable language, then $I(T, \aleph_0) \leq \aleph_0$ or $I(T, \aleph_0) = 2^{\aleph_0}$.

Also, recall that

Fact 3.2 (Vaught, 1961)**.** $I(T, \aleph_0) \neq 2$.

Fact 3.3 (Morley, 1970)**.** $I(T, \aleph_0) \in \omega \cup \{\aleph_0, \aleph_1, 2^{\aleph_0}\}$.

For linear orders with predicates there are some other results:

Theorem 3.4 (Rubin, 1974)**.** For T a theory of linear orders with unary predicates $I(T, \aleph_0) < \aleph_0$ or $I(T, \aleph_0) = 2^{\aleph_0}$, and if the language is finite then the only finite possibility is 1.

[my question: wasn't the example of a theory with exactly n models for $n \neq 2$ exactly a theory of a linear order with unary predicates?][it had constants][also, it is mentioned later]

Theorem 3.5 (Mayer, 1988)**.** For o-minimal theories, $I(T, \aleph_0) = 6^m 3^n$ or $I(T, \aleph_0) = 2^{\aleph_0}$.

Theorem 3.6 (Kulpshov, Sudoplatov, 2017)**.** For *quite o-minimal* theories, $I(T, \aleph_0) = 6^m 3^n$ or $I(T, \aleph_0) = 2^{\aleph_0}$.

Fact 3.7. For weakly o-minimal theories it does not even have to be finite. There is an example with $I(T, \aleph_0) = \aleph_0$.

Example 3.8 (Ehrenfeucht)**.** Let $L_0 = \{<, (c_n)_{n<\omega}\}$ and let T_0 say that $<$ is a DLO with no endpoints and (c_n) is increasing. Then $I(T_0, \aleph_0) = 3$.

The three models are respectively when (c_n) is unbounded, bounded with no supremum, or bounded with supremum.

To change it into an example with $n \geq 4$ countable models, let $L_n = L_0 \cup \{P_0, \dots, P_n\}$, and T_n say that T_0 holds and that $P_0, \dots, P_n$ is a dense partition and the sequence is in P_0. Here $I(T_n, \aleph_0) = n + 3$: one where the sequence is unbounded, one where it is bounded with no supremum, and for each k one where the sequence is bounded with supremum of colour k.

Let p_j be the type $\{x > c_i \mid i < \omega\} \cup \{P_j(x)\}$. Models here can be described in terms of the p_j's: if $(c_i)_{i<\omega}$ is unbounded, then $p_j(M) = \emptyset$; if $(c_i)_{i<\omega}$ is bounded with no supermum then $p_j(M) \cong \mathbb{Q}$. Finally, if $(c_i)_{i<\omega}$ is bounded with supremum coloured by P_j then $p_j(M) \cong 1+\mathbb{Q}$ and $p_k(M) \cong \mathbb{Q}$ for $k \neq j$.

Note that the theories above are not o-minimal. But we may "turn" them into o-minimal ones somehow.

Example 3.9. Let $L_n' := L_n \cup \{\prec, R_1, \dots, R_n\}$, and define T_n' to say that $\prec$ is a DLO, $P_0 \prec \dots \prec P_n$ and for each i we have $\prec\restriction P_i = <\restriction P_i$. Define $R_i(x, y)$ to hold iff $P_0(x) \wedge P_i(x) \wedge y < x$.

The idea is to forget the old order, and then it will be weakly o-minimal. T_n' can be easily axiomatised, but we will not see it.

Here we have a type $p_0 := \{P_0(x)\} \cup \{x > c_i \mid i < \omega\}$, and for $i > 0$ we have $p_i = \{P_i(x)\} \cup \{x > R_i(c_k, -) \mid k < \omega\}$.

Let $L_n'' = L_n' \setminus \{<\}$ and let T_n'' be the corresponding reduct. This is weakly o-minimal.

Definition 3.10 (Shuffling relations)**.** Let $(A, <_A)$ and $(B, <_B)$ be linear orders. We call $R \subseteq A \times B$ *shuffling* iff

1. For all $a \in A$ the set $R(a, B)$ is an initial part of B with no supremum
2. For all $a_1 <_A a_2$ we have $R(a_1, B) \subsetneq R(a_2, B)$
3. $R(A, b)$ is a final part of A with no infimum
4. $b_1 <_B b_2$ implies $R(A, b_1) \supsetneq R(A, b_2)$.

Compare with the example above, where R_i is shuffling between P_0 and P_i.

This has the following consequences

1. A, B dense
2. At most one of A and B has a minimum
3. At most one of A and B has a maximum

Example 3.11. An example where $I(T, \aleph_0) = \aleph_0$ is the following. Let

$$L = \{\prec\} \cup \{P_n \mid n \geq 0\} \cup \{R_n \mid n \geq 1\}\{c_n \mid n < \omega\}$$

Then define T to say that $\prec$ is a dlo, P_n are linearly ordered, c_n is increasing, R_n is shuffling between P_0 and P_n, and $R_n \circ R_m^{-1}$ is shuffling between P_m and P_n.

Again, we can use non-isolated types to describe models of T. We have some types $p_0, p_1, \ldots$ as in T'. Then we have a type

$$p = \{\neg P_n(x) \mid n < \omega\}$$

There are models where they are empty, they are $\mathbb{Q}$, and countably many models where all of them are $\mathbb{Q}$ except one.

Permanent Assumption 3.12. From now on, assume T is weakly o-minimal.

Fact 3.13. If $p \in S_1(\emptyset)$ is non-algebraic, then there are exactly two global $\emptyset$-invariant extensions of p, which we call p_ℓ and p_r: one "at the left" of the original locus, and one "at the right".

Definition 3.14. Fix $p \in S_1(\emptyset)$. For $a \vDash p$, define

$$\begin{aligned} \Sigma_p(a) &:= \{b \vDash p \mid b \not{m}\, odelsp_r \restriction a, a \not{m}\, odelsp_r \restriction b\} \\ &(= \{b \vDash p \mid b \not{m}\, odelsp_\ell \restriction a, a \not{m}\, odelsp_\ell \restriction b\}) \\ &(= \{b \vDash p \mid b \in \varphi(a, \mathfrak{U})\text{-bounded in } p(\mathfrak{U})\}) \end{aligned}$$

Proposition 3.15. "$x \in \Sigma_p(y)$" is an equivalence relation on $p(\mathbb{C})$ with convex classes. It is $\bigvee$-definable.

Definition 3.16. Let Σ_p^M be the restriction of Σ_p on $p(M)$. The order type of $p(M)/\Sigma_p^M$ is denoted $\mathrm{Inv}_p(M)$.

Let p^n be the type of a Morley sequence in p_r over $\emptyset$ of length n.

Remark 3.17. $\Sigma_p(a) < \Sigma_p(b)$ iff $(a, b) \vDash p^2$.

Definition 3.18. p is *trivial* for every n we have

$$p^2(x_1, x_2) \cup \ldots \cup p^2(x_{n-1}, x_n) \vdash p^n(x_1, \ldots, x_n)$$

Remark 3.19. In the binary case, i.e. when every formula is a Boolean combination of formulas in two free variables, every p is trivial.

Definition 3.20. Say that p is *simple* iff $\Sigma_p(a)$ is a-definable.

Theorem 3.21. If p is trivial and non-simple, then the invariant $\mathrm{Inv}_p(M)$ can be any countable linear order.

Corollary 3.22. In the binary case, if p is non-simple, then $I(T, \aleph_0) = 2^{\aleph_0}$.

Theorem 3.23. If p is simple, then $\mathrm{Inv}_p(M)$ is a dense linear order, possibly with endpoints.

Corollary 3.24. Possibilities are $\emptyset$, 1, $\mathbb{Q}$, $1 + \mathbb{Q}$, $\mathbb{Q} + 1$, $1 + \mathbb{Q} + 1$.

The above is where the 6 comes from in the o-minimal case: you multiply by the number of cuts [or something like that].

The main difference between o-minimal and weakly o-minimal theories is the following. Recall that

Definition 3.25. $p \perp^w_q$ (p and q are *weakly orthogonal*) iff $p(x) \cup q(y)$ has only one extension.

We say that $p \not\perp^w_b$ iff there is $\theta(x, y)$ such that $\theta(a, \mathfrak{U})$ is bounded in $q(\mathfrak{U})$ for[1] $a \models q$.

We say $p \not\perp^w_u$ (unbounded) otherwise.

Remark 3.26. $\not\perp$ and $\not\perp^w_b$ are equivalence relations.

The difference is that in o-minimal theories all non-orthogonalities are bounded.

Remark 3.27. $p \not\perp^w_u$ iff $p(x) \cup q(y)$ has 2 extension.

Theorem 3.28. if $p \not\perp^w_b q$, then $\mathrm{Inv}_p(M)$ is isomorphic to $\mathrm{Inv}_q(M)$ or to the revers of it. Also, the isomorphism is canonical, in the sense that it does not depend on the choice of θ.

Theorem 3.29. If $p \not\perp^w_u q$, then there is a definable shuffling relation between $p(M)$ and $q(M)$.

Remark 3.30. Isolated types are weakly orthogonal to non-isolated types. The former are any fixed in the models, so we can just ignore them.

Theorem 3.31. In the binary case, if $I(T, \aleph_0) < 2^{\aleph_0}$ then $\not\perp^w$ has finitely many classes among the non-isolated types.

Choose representatives of $\not\perp^w$-relations $p_1, \ldots, p_n$. In $[p_i]_{\not\perp^w}$ choose representatives of $\not\perp^w_b$, say $\{p_{ij} \mid j < U_i\}$, for some $U_i \leq \omega$.

$$\mathrm{Inv}(M) = \mathrm{Inv_{p_{ij}}(M) \mid 1 \leq i \leq n, 1 \leq j \leq U_i}$$

Theorem 3.32. $\mathrm{Inv}(M) = \mathrm{Inv}(N)$ if and only if $M \cong N$.

Proof Idea. Consider Morley sequences for each of the types, and models of the chosen sequences are prime, so you can use back an forth. □

This of course implies Vaught's conjecture in the case at hand.

[1] Recall that we are in weakly o-minimal theories and loci are convex.

3.2 Gabriel Conant – Stability and sparsity in sets of natural numbers

We are interested in expansions of $(\mathbb{Z}, +, 0)$.

Example 3.33. $(\mathbb{Z}, +, \cdot, 0)$ is very bad: it has all "untameness" properties you can imagine. Note that $<$ is definable (e.g. with the four squares theorem).

Example 3.34. In $(\mathbb{Z}, +, 0, P)$, where P is the set of primes, assuming Dickson's Conjecture[2] you can define the multiplication.

Example 3.35. $(\mathbb{Z}, +, 0, f(\mathbb{N}))$, where $f(X) \in \mathbb{Z}[X]$. Here if $\deg(f) \geq 1$ the ordering is definable. In fact, if $\deg(f) \geq 2$ then multiplication is definable (not trivial).

Example 3.36. $(\mathbb{Z}, +, <, 0)$, a.k.a. Presburger arithmetic is unstable, but it eliminates quantifiers in a suitable language and there is a nice cell decomposition (reminiscent of o-minimality) for definable sets. It is NIP, even dp-minimal.

Example 3.37. $(\mathbb{Z}, +, 0)$ is very (almost too) nice: it is a stable group of U-rank 1. A set $X \subseteq \mathbb{Z}^n$ is definable (with parameters) if and only if it is a finite Boolean combination of cosets of subgroups of $\mathbb{Z}^n$. Still, it is not ω-stable.

The structure above has been understood for a long time, but still there are some questions.

Question 3.38 (Marker, 2011)**.** Are there any proper stable expansions of $(\mathbb{Z}, +, 0)$?

(it is somehow connected to the free group). The answer came some years later, in 2014, by Poizat and independently by Palacin and Sklinos:

Fact 3.39. If $A = \{q^n \mid n \in \mathbb{N}\}$ for some $q \geq 2$, then $(\mathbb{Z}, +, 0, A)$ is superstable of U-rank ω.

Fact 3.40 (Palacin-Sklinos)**.** Same for $A = \{n! \mid n \in \mathbb{N}\}$ and $A = \left\{ q_1^{q_2^{\cdot^{\cdot^{\cdot^{q_t^n}}}}} \mid n \in \mathbb{N} \right\}$.

Question 3.41. Can one characterise these sets A?

We are going to see three results. The first is

Theorem 3.42. Let Γ be a finitely generated multiplicative submonoid of $\mathbb{Z}^+$. Then, for any infinite $A \subseteq \Gamma$, the structure $(\mathbb{Z}, +, 0, A)$ is superstable of U-rank ω.

[2] It is about primes in arithmetic progressions.

For the other two we need some definitions.

Definition 3.43. A sequence $(\lambda_n)_{n=0}^{\infty} \subseteq \mathbb{R}^+$ is *geometric* iff it is strictly increasing and $\{\lambda_n/\lambda_m \mid m \le n\}$ is a closed, discrete set.

Example 3.44. Some example of geometric sequences:

- $(\tau^n)_{n=0}^{\infty}$ where $\tau > 1$.
- Any $(\lambda_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \lambda_{n+1}/\lambda_n = \infty$
- Any $(\lambda_n)_{n=0}^{\infty}$ such that $\forall n \in \mathbb{N}\ \lambda_{n+1}/\lambda_n \in \mathbb{Z}^+$

Remark 3.45. Look at the A's in the examples before; they [most of them?] are of that kind.

Theorem 3.46. Suppose $(\lambda_n)_{n=0}^{\infty} \subseteq \mathbb{R}^+$ is geometric and $B = (b_n)_{n=0}^{\infty} \subseteq \mathbb{Z}$ is such that $|b_n - \lambda_n|$ is $\mathcal{O}(1)$. Then for any finite $F \subseteq \mathbb{Z}$ and any infinite $A \subseteq B + F$, then $(\mathbb{Z}, +, 0, A)$ is superstable of U-rank ω.

Example 3.47. If $(b_n)_{n=0}^{\infty}$ is the Fibonacci sequence, take $\lambda_n = \varphi^n\sqrt{5}$, where φ is the Golden ratio.

[my question: any relation with stuff like being piecewise syndetic? with topological dynamics?][well, a linear polynomial...]

Question 3.48. How much can $\mathcal{O}(1)$ be weakened?

Theorem 3.49. Suppose $(\lambda_n)_{n=0}^{\infty} \subseteq \mathbb{R}^+$ is geometric and $B = (b_n)_{n=0}^{\infty} \subseteq \mathbb{Z}^+$ is such that $|b_n - \lambda_n|$ is $o(\lambda_n)$ and $\{\lambda_n \mid n \in \mathbb{N}\}$ is linearly independent over $\mathbb{Q}$. Same conclusion of Theorem 3.46.

Example 3.50. $A = \{\lfloor \tau^n + f(n) \rfloor \mid n \in \mathbb{N}\}$, where $\tau > 1$ is transcendental and $f(n)$ is $o(\tau^n)$. For instance $\lfloor \pi^n + n^2 + \log n \rfloor$.

Definition 3.51. Fix $A \subseteq \mathbb{Z}$.

1. Given $n > 0$, let $\Sigma_n(A) := \{a\}_1 + \ldots + a_k \mid k \le n, a_1, \ldots, a_k \in A$ (the a_i are not necessarily distinct).
2. Let A^{ind} be the structure with universe A and, for all $X \subseteq \mathbb{Z}^n$ $\emptyset$-definable in $(\mathbb{Z}, +, 0)$, a relation symbol for $X \cap A$.

Of course in order for $(\mathbb{Z}, +, 0, A)$ to be stable, A^{ind} must be stable. There is some work on the converse, in general. We will use this specialisation:

Lemma 3.52. Fix $A \subseteq \mathbb{Z}$. Suppose that for all $n > 0$ the set $\Sigma_n(\pm A)$ does not contain a non-trivial subgroup of $\mathbb{Z}$. Then:

a) $\text{Th}(\mathbb{Z}, +, 0, A)$ is stable if and only if $\text{Th}(A^{\text{ind}})$ is stable.

b) $\omega \leq U(\mathbb{Z}, +, 0, A) \leq U(A^{\text{ind}}) \cdot \omega$ (ordinal multiplication)

(recall that by Palacin-Sklinos there are no proper stable expansion of finite U-rank)

Remark 3.53. A's like above are automatically not definable in $(\mathbb{Z}, +, 0)$, by the characterisation in terms of cosets.

Consider $A \subseteq \mathbb{Z}$. Sometimes it is enough to look to a reduct of A^{ind}.

Definition 3.54. $A_0^{\text{ind}} := (A, (\{\varepsilon_1 x_1 + \ldots + \varepsilon_k = 0\} \cap A^k)_{k>0, \varepsilon_i \in \{1,-1\}})$.

Basically, we are throwing away congruences.

Proposition 3.55. A^{ind} is (a definable expansion of) the expansion of A_0^{ind} by unary predicates for $A(\cap n\mathbb{Z} + r)$ for $0 \leq r < n$.

Definition 3.56. A structure M is *monadically stable* if the expansion of M by all predicates for subsets of M^1 is stable.

The task now is: for $A \subseteq \mathbb{Z}$ as in Theorem 3.42, Theorem 3.46 or Theorem 3.49, show:

(i) $\forall n > 0$ $\sigma_n(A)$ does not contain a nontrivial subgroup of $\mathbb{Z}$

(ii) A_0^{ind} is monadically stable (and the monadic expansion has U-rank 1).

For Theorem 3.42, one uses the following

Fact 3.57 (Evertse, SChlickewei, Schmidt 2002)**.** Let G be a finitely generated multiplicative subgroup of $\mathbb{C}^\times$. For any $k > 0$ there is $C = C(G, k)$ such that for all $\alpha_1, \ldots, \alpha_k, \beta \in \mathbb{C}^\times$, the equation $\sum_{i=1}^k alpha_i x_i = \beta$ has $\leq C$ solutions in G^k such that $\sum_{i \in I} a_i x_i \neq 0$ for $\emptyset \neq I \subseteq \{1, \ldots, k\}$.

(the proof follows from a large body of work) Given this, one proves even something stronger than task (i) no arbitrarily long arithmetic progressions (as opposed to subgroups).

For task (ii), wlog (modulo some work) A is a finitely generated multiplicative submonoid generated by $q_1, \ldots, q_d$ such that $\{\log q_1, \ldots, q_d\}$ are linearly independent over $\mathbb{Q}$. Then by the Fact above we get that A_0^{ind} is a reduct[3] of $(\mathbb{N}^d, s_1, \ldots, s_d)$, where s_i is the successor on the $i$þcoordinate.

[3] $q_1^{n_1} \cdots q_d^{n_t} \mapsto (n_1, \ldots, n_d)$.

Chapter 4

Monday 3 July

4.1 Ludomir Newelski – Topological dynamics of stable groups

[slides]

Let G be a stable group in a countable language L, and let T be its theory. Let G^* be a monster for it.

Question 4.1. Can we do stable group theory without forking?

The idea is that a positive answer may lead to generalisations outside of stable theories.

Definition 4.2. Let $p, q \in S(G)$. Define $p * q = \mathrm{tp}(a \cdot b/M)$, where $a \models p$, $b \models q$ and $a \mathop{\smile\!\!\!\!\!\mid}_G b$.

Fact 4.3. $(S(G), *)$ is a semigroup.

[G-flow] With "point-transitive" we mean there is a dense G-orbit (G-ambit). This last hypothesis can be relaxed.

Let X be a G-ambit. [enveloping semigroup $E(X)$]

Sometimes $X \cong E(X)$. Let $\mathcal{A} \subseteq \mathscr{P}(G)$ be a G-algebra of sets (i.e. closed under left translations). Then $S(\mathcal{A})$ is a G-flow. For $p \in S(\mathcal{A})$, define $d_p \colon A \to \mathscr{P}(G)$ by

$$d_p(U) = \{g \in G \mid g^{-1}U \in p\}$$

Definition 4.4. $\mathcal{A}$ is *d-closed* iff $\mathcal{A}$ is closed under d_p for every $p \in S(\mathcal{A})$.

Example 4.5. (T stable) $\mathcal{A} = \mathrm{Def}(G)$ is d-closed, because every $p \in S(G)$ is definable.

Fact 4.6. Assume that $\mathcal{A}$ is d-closed.

- For $p \in S(\mathcal{A})$, d_p is a G-endomorphism of $\mathcal{A}$.

- The map $d\colon S(\mathcal{A}) \to \mathrm{End}(\mathcal{A})$ sending p to d_p is a bijection.
- d induces $*$ on $S(\mathcal{A})$ so that $d(S(\mathcal{A}), *) \cong (\mathrm{End}(\mathcal{A}), \circ)$.

Theorem 4.7. $E(S(\mathcal{A})), \circ \cong^1 (S(\mathcal{A}), *) \cong^2 (\mathrm{End}(A), \circ)$.

Proof. For $p \in S(\mathcal{A})$, let $l_p(q) = p * q$. Then $l_p \in E(S(\mathcal{A}))$ and $p \mapsto l_p$ gives $\cong^1$. As for $\cong^2$, this is d. □

Example 4.8. (T stable) If $\mathcal{A} = \mathrm{Def}(G)$ then $\mathcal{A}$ is d-closed, and $S_G(M) = S(\mathcal{A})$ from the Theorem above is just the free multiplication of G-types mentioned earlier.

Definition 4.9. $\Delta \subseteq L$ is *invariant* iff the family of relatively Δ-definable subsets of G is closed under left and right translation in G, and also under taking inverses. Let $\mathrm{Inv} := \{\Delta \subseteq_{\mathrm{fin}} L \mid \Delta \text{ is invariant}\}$.

Fact 4.10. Inv is cofinal in $[L]^{<\omega}$, i.e. every finite subset of L is contained in some invariant subset.

Fix $\Delta \in \mathrm{Inv}$. Let $\mathrm{Def}_\Delta(G)$ be the set of relatively Δ-definable subsets of G, and let $S_\Delta(G)$ be the associate Stone space.

Fact 4.11.

1. $\mathrm{Def}_{G,\Delta}(M)$ is a d-closed G-algebra of sets (this relies on the full definability lemma in local stability)
2. By the Theorem above, $(S_{G,\Delta}(M), *) \cong E(S_{G,\Delta}(M), \circ) \cong \mathrm{End}\,\mathrm{Def}_{G,\Delta}(M), \circ$
3. $\mathrm{Def}_G(M) = \bigcup_{\Delta \in \mathrm{inv}} \mathrm{Def}_{G,\Delta}(M)$
4. $\langle S_{G,\Delta}(M), \Delta \in \mathrm{Inv}\rangle$ is an inverse system of G-flows and semigroups (the connecting functions are the restrictions)
5. $S_G(M) = \mathrm{invlim}_{\Delta \in \mathrm{Inv}} S_{G,\Delta}(M)$, both as G-flows and as semigroups.

Now we considered types as functions. The motivation comes from algebraic geometry. When you look at an algebraic variety, you speak of the regular functions (or rational ones). So here we look as types as functions by sending p to $d_p\colon \mathrm{Def}(G) \to \mathrm{Def}(G)$. The same thing can be done locally, i.e. from $S_\Delta(G)$ to $\mathrm{Def}_\Delta(G) \to \mathrm{Def}_\Delta(G)$.

Now, d_p has a kernel and an image. The kernel has a clear topological dynamics mean:

$$\mathrm{Ker}\, d_p = \{U \in \mathrm{Def}_\Delta(G) \mid [U] \cap \mathrm{cl}(Gp) = \emptyset\}$$

The idea is:

- the larger the type $p \in S_G(M)$ (or $S_{G,\Delta}(M)$)

- the smaller the flow $\mathrm{cl}(Gp)$ (a large type is a type with a small orbit; e.g. a generic type as a bounded orbit, while a realised one…)
- the larger the kernel $\operatorname{Ker} d_p$
- the smaller the image
- the larger the (local) Morley rank of p

So these are all ways of measuring how large a type is.

Let p^{*n} be $p * \ldots * p$ (n times), so $d_{p^{*n}} = d_p \circ \ldots \circ d_p$. Let $R(p) = \langle \mathrm{MR}_\Delta(p) \mid \Delta \in \mathrm{Inv}\rangle$.

Lemma 4.12. *$R(p^{*n})$ grow (coordinatewise), $\mathrm{Ker}(d_{p^{*n}})$ and $\mathrm{Im}(d_{p^{*n}})$ shrink as n increases. Moreover, the growth/shrinking of these three sequences is strictly correlated.*

There are similar connections between MR_Δ, Ker and Im in $S_\Delta(G)$. In particular, if $\mathrm{MR}_\Delta(U) < \mathrm{MR}_\Delta(p)$, then [$U \in \operatorname{Ker} d_p$?]

Assume $\mathcal{A} \subseteq \mathrm{Def}_\Delta(G)$ is a G-subalgebra.

$\mathcal{A}$ is scattered, has finite CB-rank, MR_Δ-rank. $\mathcal{A}$ is atomic; for $g \in G$, let $U_g \in \mathcal{A}$ be the atm containing g. This always exists because $\mathcal{A}$ is a G-algebra: since $\mathcal{A}$ is atomic there is some atom $U = U_h \in \mathcal{A}$. But $\mathcal{A}$ is closed under G-translations: $1 \in h^{-1}U_h = U_1$, and $g \in gU_1 = U_g$. Also, U_1 is a (definable!) subgroup of G, and $U_g = gU_1$ is the left coset of U_1 containing g.

Atoms almost determine $\mathcal{A}$. Let $p \in S_\Delta(G)$ and $\mathcal{A} = \mathrm{Im}(d_p)$.

Let $p \in S_\Delta(G)$ and $\mathcal{A} = \mathrm{Im}(d_p)$. What is U_1? For $V \in \mathrm{Def}_\Delta(G)$, we have $1 \in d_p(V) \iff V \in p$. Choose $V \in p$ with $\mathrm{CB}(V) = \mathrm{CB}(p)$ and $\mathrm{Mlt}(V) = \mathrm{Mlt}(p)$ (ranks evaluated in $\mathrm{Def}_\Delta(G)$. Then $U_1 = d_p(V)$ [more stuff I missed]

Assume $p \in S(G)$, and G^* is a monster. $p(G^*)$ generates a sugroup $\langle p(G^*)\rangle < G^*$, invariant under $\mathrm{Aut}(G^*/G)$. Let $\langle p\rangle$ be the minimal type-definable subgroup of G^* containing $\langle p(G^*)\rangle$. Let $\mathrm{Cl}_*(p) = \mathrm{cl}(\{p^{*n} \mid n \in \mathbb{N}\})$. So $p^n(G^*) \subseteq \langle p\rangle$.

Theorem 4.13. *The generic types of $\langle p\rangle$ are precisely the types in $\mathrm{Cl}_*(p)$ with maximal ranks $\mathrm{MR}_\Delta, \Delta \subseteq L$ finite, invariant.*

$\mathrm{rmor}_\Delta(p_0 * p_1) \geq \mathrm{MR}_\Delta(p_i)$. So $(\mathrm{MR}_\Delta(p^n))_n$ is non-decreasing.

If $q = \lim_i p * n_i$, then $\mathrm{MR}_\Delta(q) \geq \mathrm{MR}_\Delta(p^{n_i})$. If $q \in \mathrm{Cl}_*(p)$ is a generic type of $\langle p\rangle$, then q is an accumulation point of $\{p^n \mid n > 0\}$. The converse of this is not necessarily true. But almost:

Theorem 4.14 (2-step Theorem)**.** *Let q be an accumulation point of $\{p^n \mid n > 0\}$ and r an accumulation point of $\{q^n \mid n > 0\}$. Then r is a generic type of $\langle p\rangle$.*

The theorem above is slightly mysterious. It seems to be related to the growth rate of some ranks.

Proposition 4.15. Let $p \in S(G)$ or $S_\Delta(G)$ and $q = p^{-1} * p$. Then the group $\langle q \rangle$ is connected, i.e. there is only one generic type.

This is classical, but we will see another proof with no reference to forking.

Proof. WLOG $p \in S_\Delta(G)$, so $q \in S_\Delta(G)$. We have a chain

$$\mathrm{Im}(d_{q^1}) \supseteq \mathrm{Im}(d_{q^2}) \supseteq \mathrm{Im}(d_{q^3}) \supseteq \dots$$

Using the CB-rank one can show there is n such that $\mathrm{Im}(d_{q^n}) = \mathrm{Im}(d_{q^{n+k}})$ for all k. Let $\mathcal{A} = \mathrm{Im}(d_{q^n})$. So $d_{q^{n+1}} = d_{p^{-1}} \circ d_p \circ d_{q^n}$ We shall prove that $d_q \restriction \mathcal{A} = \mathrm{id}_{\mathcal{A}}$. First of all, it is injective, so $d_p \colon \mathcal{A} \cong \mathcal{A}' \subseteq \mathrm{Def}_\Delta(G)$. To see this injectivity, assume not and notice that the CB-rank would have to go down but $d_{q^{n+1}} = d_{p^{-1}} \circ d_p \circ d_{q^n}$ have the same images and by definition of n the rank has stabilised. Since $d_p(U_1) \neq \emptyset$, we have $U_s = sU_1 \in p$ for some $s \in G$.

$$U_1' = sU_1 s^{-1} = U_1^s = d_p(U_s)$$

$d_p(U_h) = d_p(hs^{-1}U_s) = hs^{-1} d_p(U_s) = hs^{-1}U_1' = U'_{hs^{-1}}$. Since $\mathrm{Im}(d_{q^{n+1}}) = \mathrm{Im}(d_{q^n})$, we have that $d_{p^{-1}} \colon \mathcal{A}' \cong \mathcal{A}$. Since $d_{p^{-1}}(U_1') \neq'$ we have that, for some $t \in G$, $U_t' = tU_1' \in p^{-1}$. We can take $t = s^{-1}$, because $sU_1 \in p \Rightarrow (sU_1)^{-1} = U_1 s^{-1} \in p^{-1}$, and $U_1 s^{-1} = s^{-1} s U_1 s^{-1} = s^{-1} U_1'$. Now we have that $d_q(U_1) = U_1$[calculation omitted]. So $d_q(U_h) = U_h$ for all h, and this means that d_q is the identity on $\mathcal{A}$. Hence $q^n * q^n = q^n$ and q^n is the generic type of U_1. So $U_1 = \langle q \rangle$. □

This was an example of an argument avoiding forking. Hence the hope is that it will be possible to apply these techniques in unstable theories.

4.2 Philipp Hieronymi – On continuous functions definable in expansions of the ordered real additive group

[slides] [joint with Walsberg and Fornasiero]
We consider expansions $\mathcal{R}$ of $(\mathbb{R}, <, +)$. The motivation is

Question 4.16. Can we classify such expansions according to the geometric/topological complexity of its definable sets (definable functions)?

There are two perspectives on first-order expansions of $(R, <, +)$:

1. As a concrete collection of (definable) subsets of $\mathbb{R}^n$
2. In terms of its theory, up to bi-interpretability.

In this talk we take perspective 1.

Definition 4.17. An *ω-orderable set* is a definable set that admits a definable ordering with order type ω. We say that such a set is *dense* if it is dense in some open subinterval of $\mathbb{R}$.

Fact 4.18 (Trivial trichotomy)**.** An expansion $\mathcal{R}$ of $(\mathbb{R}, +, <)$ satisfies exactly one of the following three conditions:

(A) does not define a dense ω-orderable set

(B) defines a dense ω-orderable, but avoids a compact set

(C) defines every compact set.

Remark 4.19. In case (C) there is no model-theoretic tameness. Such structures are best dealt with descriptive set theory.

Remark 4.20. O-minimality implies that we are in case (A). Actually, NTP_2 already implies it. Type (B) is best dealt with tame geometry.

Remark 4.21. Type (B) interprets $(\mathbb{N}, \mathscr{P}(\mathbb{N}), +1, \in)$. Can be decidable. Type (B) is best dealt with automata.

Remark 4.22. Type (A) can interpret $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$. So structures there can still be not tame.

[quote by Shelah on decidability: as biological taxonomy does not tell whether a species is tasty, the classification does not deal with decidability]

Remark 4.23. If D is ω-orderable, so is D^k. If D is ω-orderable and $f \colon D \to \mathbb{R}$ is definable, then $f(D)$ is ω-orderable (if $f(D)$ is infinite).

Remark 4.24. if $D \subseteq \mathbb{R}$ is infinite, closed and discrete, then D is ω-orderable (but obviously not dense)

Remark 4.25. If $D \subseteq \mathbb{R}$ is a closed and discrete set and $f \colon D^n \to \mathbb{R}$ is such that $f(D^n)$ is somewhere dense, then $f(D^n)$ is a dense ω-orderable set.

[we do not really care about density for $n > 1$, so we do not define it]

Theorem 4.26. Let $\mathcal{R}$ be an expansion of the real field. If $\mathcal{R}$ defines a dense ω-orderable set, then $\mathcal{R}$ defines $\mathbb{Z}$ and hence $\mathcal{R}$ defines every closed set.

Corollary 4.27. In this case it cannot be type (B).

For $r \in \mathbb{N}_{\geq 2}$, consider a ternary predicate $V_r(x, u, k)$ that holds if and only if u is an integer power of r, $k \in \{0, \dots, r-1\}$, and the digit of some base r representation of x in the position corresponding to u is k. Set $\mathcal{T}_r \coloneqq (\mathbb{R}, <, +, V_r)$.

Remark 4.28. $\mathcal{T}_r$ is bi-interpretable with $(\mathbb{N}, \mathscr{P}(\mathbb{N}), +1, \in)$.

Let D_r be the set of numbers in $[0, 1)$ admitting a finite base r expansion. This is clearly dense in the interval. Define $\tau_r : D_r \to r^{-\mathbb{N}_{>0}}$ so that $\tau_r(d)$ is the elast $u \in r^{-\mathbb{N}_{>0}}$ appearing with nonzero coefficient in the finite base r expansion of d.

For $d, e \in D_r$, let $d \prec_r e$ iff $\tau_r(d) > \tau_r(e)$ or $\tau_r(d) = \tau_r(e) and d < e$.

Remark 4.29. $(D_r, \prec)$ is definable in $\mathcal{T}_r$ and has order type ω.

Example 4.30. Two more natural but non-obvious examples of type (B) structures are:

- $(\mathbb{R}, <, +, C)$, where C is the middle-thirds Cantor set
- $(\mathbb{R}, <, +, \mathbb{Z}, x \mapsto \alpha x)$, where α is irrational and quadratic.

Both of them are bi-interpretable with $(\mathbb{N}, \mathscr{P}(\mathbb{N}), +1, \in)$.

Question 4.31 (Open)**.** Is this true for every type (B) structure?

It is very hard to expand type (B) structures without becoming type (C): $(\mathbb{R}, <, +, \mathbb{Z}, x \mapsto \alpha x, C)$ is type (C). With α irrational and non-quadratic, $(\mathbb{R}, <, +, \mathbb{Z}, x \mapsto \alpha x)$ is type type (C). This phenomenon has parallels in automata theory.

What about definable functions?

Remark 4.32. In type (C) every continuous function $f : [0, 1] \to \mathbb{R}$ is definable (its graph is a compact set)

Fact 4.33. In type (A), let $f : [0, 1] \to \mathbb{R}$ be a definable continuous function on an open interval I. Then for every $k \in \mathbb{N}$ there is a definable open dense $U \subseteq I$ on which f is C^k.

(note the analogy with o-minimality and recall that every NTP_2 structure is type (A)) The proof uses an idea of Laskowski and Steinhord (to use a Theorem of Boas-Widder).

Fact 4.34. In type (B) every definable C^2 function $f : [0, 1] \to \mathbb{R}$ is affine.

For the proof use results of Hieronymi and Tyconievich.

These arguments work for *continuous* functions. You can stay in type (A) even with functions with dense graphs.

Let $C^\infty([0, 1])$ be the space of smooth functions with the topology induced by the semi-norms $f \mapsto \max_{t\in[0,1]} |f^{(j)(t)}|$ for $j \in \mathbb{N}$.

Fact 4.35 (Grigoriev 2005, LeGal 2009)**.** The set of all $f \in C^\infty[0, 1]$ uch that $(\mathbb{R}, +, <, f)$ is o-minimal is co-meager.

Theorem 4.36. The set of all $f \in C^k[0,1]$ such that $(\mathbb{R}, <, +, f)$ is type C is comeager in $C^k([0,1])$. (use similarly defined seminorms)

Proof. The somewhere $(k+1)$-differentialbe functions in C^k are meager, so no type (A). But type (B) stuff is all affine. ☐

Definition 4.37. Say that $\mathcal{R}$ is *of field-type* iff there is an interval $I \subseteq \mathbb{R}$ together with definable functions $\oplus, \otimes \colon I^2 \to I$ suc that $(I, <, \oplus, \otimes)$ is isomorphic to $(\mathbb{R}, <, +, \cdot)$.

Theorem 4.38 (Marker, Pillay, Peterzil 1992)**.** Suppose $\mathcal{R}$ is o-minimal. If $\mathcal{R}$ defines a non-affine C^1 function $[0,1] \to \mathbb{R}$, then $\mathcal{R}$ is of field-type.

Theorem 4.39. Suppose $\mathcal{R}$ defines a non-affine C^1 function $[0,1] \to \mathbb{R}$. Then (at least[1]) one of the following holds:

1. $\mathcal{R}$ is of field-type (in particular, when f is C^2)
2. $\mathcal{R}$ interprets $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$.

Corollary 4.40. Suppose that $\mathcal{R}$ is type (A). Then one of the following holds:

1. $\mathcal{R}$ is of field-type
2. FOr every continuous definable $f \colon [0,1] \to \mathbb{R}$ there is an open dense $U \subseteq [0,1]$ such that f is affine on each conencted component of U.

Corollary 4.41. [missed]

[proof idea]

Fact 4.42 (I may have missed some blanket hypotheses. But probably not.)**.** Let $F \subseteq \mathbb{R}$ be nowehere dense and $\oplus$, $\otimes$ be definable functions such that $(F, <, \oplus, \otimes)$ is an ordered field isomorphic to $(\mathbb{R}, <, +, \cdot)$. Then there is a definable $Z \subseteq F$ such that $(F, <, \oplus, \otimes, Z)$ is isomorphic to $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$.

[proof sketch] An application to metric dimensions is the following:

Theorem 4.43 (Hieronymi, Miller 2015)**.** Let $\mathcal{R}$ be an expansion of $(\mathbb{R}, <, +, \cdot)$ that is not type (C), then the Assouad dimension of any compact definable subset of $\mathbb{R}^n$ agrees with its topological dimension.

Remark 4.44. All commonly encountered notions of metric dimension are bounded from below by the topological one and above from the Assouad one. So no fractals!

Remark 4.45. Fails without multiplication. For example $(\mathbb{R}, <, +, C)$ is type (B).

[1] Not clear if it is exclusive.

Theorem 4.46. The Assouad dimension of any compact definable subset of $\mathbb{R}^n$ agrees with the topological one if one of the following holds:

1. $\mathcal{R}$ defines a non-affine C^1 function and does not interpret $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$.

2. $\mathcal{R}$ defines a non-affine C^2 function and is not type (C).

4.3 Katrin Tent – Complexity of Topological Group Isomorphism

[joint with Kechris and Nies]
The motivation is the following

Fact 4.47. ω-categorical structures are biinterpretable if and only if their automorphism groups are isomorphic as topological groups.

Recall that

Fact 4.48. If M is a countable structure, then $\mathrm{Aut}(M)$ is a closed subgroup of S_∞. Conversely, every closed subgroups of S_∞ arise that way.

The topology is the product topology/pointwise convergence topology. One way of specifying this is by saying that a neighbourhood basis for the identity is given, for $A \in \mathscr{P}_{\mathrm{fin}}(\mathbb{N})$, by the pointwise stabilisers $N_A = \{g \in S_\infty \mid g \upharpoonright A = \mathrm{id}_A\}$.

Fact 4.49. With this topology, S_∞ is a Polish group. As a neighbourood basis is given by open subgroups, the group is totally disconnected.

Question 4.50. Given a class $\mathcal{C}$ of closed subgroups of S_∞, how difficult it is to determine whether two of them are isomorphic as topological groups?

A natural way of measuring "how difficult" is by means of *Borel reducibility.*

Definition 4.51. Let X, Y be standard Borel spaces, i.e. Polish spaces with the Borel σ-algebra. A map $X \to Y$ is called *Borel* iff all preimages of Borel subsets are Borel.

Example 4.52. $\mathbb{R}$, 2^ω are standard Borel spaces. We will be interested in the following: fix L a countable relational language. Then the class of all countable L-structures is also a standard Borel space.

To see this, put $X_L = \prod_{i \in I} \mathscr{P}(\mathbb{N}^{n_i})$ where $L = (R_i)_{i \in I}$ and R_i has arity n_i.

Definition 4.53. Let E, F be equivalence relations on X and Y respectively. We say that $E \leq_B F$ (*E is Borel-reducible to F*) iff there is a Borel function $g \colon X \to Y$ such that $uEv \iff g(u)Fg(v)$. We write $E \equiv_B F$ iff $E \leq_B \leq F \leq_F E$.

Example 4.54. We now see some "benchmark" equivalence relations one usually looks at to know how difficult something else is.

- $\mathrm{id}_{\omega^\omega} \equiv_B \mathrm{id}_{2^\omega}$
- E_0, i.e. "almost equality" on ${}^2\omega$, which is Borel-equivalent to isomorphism of subgroups of $(\mathbb{Q}, +)$, i.e. rank 1 torsion-free groups (by a result of [Baire?]).
- GI, which is isomorphism on countable graphs. This is Borel-equivalent to isomorphism on all L-structures, for countable, relational L.

We have $\mathrm{id}_{2^\omega} <_B E_0 < \mathrm{GI}$. There is nothing between the first two, but infinitely many things between the second to.

How do we consider closed subgroups of S_∞ as a Borel space? A standard procedure in descriptive set theory is this one:

Definition 4.55. If X is a Polish space, the *Effros space* $F(X)$ is the set of all closed subsets of X with this topology. For $U \subseteq X$ open, put in a basic open set $B_U = \{D \in F(X) \mid D \cap U \neq \emptyset\}$. [This is again a Polish space; maybe anyway we are only interested in the σ-algebra generated] Then take the Borel σ-algebra.

The idea is to see the closed subgroups as points in Effros space. So we consider the Borel subclasses of $F(S_\infty)$ consisting of subgroups.

Example 4.56. We are interested in the following ones (one has to show they are Borel, it is not generally trivial):

1. Compact (=profinte); all orbits are finite
2. Locally compact
3. Oligomorphic (=ω-categorical)
4. Roelulke precompact[2]

What about the isomorphism problem for finitely generated profinte groups?

Theorem 4.57 (Jarden-Lubotzy)**.** If G, H are topologically finitely generated[3] profinite groups, then G and H are topologically isomorphic if and only if G, H have the same finite quotients.

[2]A generalisation of compactness. By a result of Tanovic, equivalently they are inverse limits of oligomorphic groups. Equivalently, automorphism groups of ω-categorical *metric* structures.

[3]I.e. there is a dense finitely generated group.

(a corollary also says that if they are elementarily equivalent they as groups they are isomorphic) This says that you can find a map from that class to 2^ω and you put 1 if the corresponding finite groups comes out, otherwise you put 0. This shows reducibility to id_{2^ω} [my question: why is that Borel?]. To show Borel equivalence one has to show that there are uncountably many isomorphism types. Anyway,

Fact 4.58 (Lubotzky). Put $G_P = \prod_{p\in P} \mathrm{SL}_2(\mathbb{Z}_p) \cong \mathrm{SL}_2(\hat{\mathbb{Z}})/\prod_{q\notin P}\prod \mathrm{SL}_2(\mathbb{Z}_q)$, where P is a set of primes. Two such are isomorphic if and only if P is the same.

Passing from finitely generated to profinite makes the complexity go up: one can show that the graph isomorphism is reducible:

Proposition 4.59. GI is Borel-reducible to the isomorphism problem on profinite subgroups of S_∞. More specifically, even to profinite 2-step nilpotent groups of exponent $p \geq 3$.

Essentially due to Mekler. Mekler showed how to code finite graphs in groups. So code graphs in such groups. □

Theorem 4.60. Let $\mathcal{C}$ be a Borel class of closed subgroups of S_∞ closed under conjugation in S_∞. Assume that for $G \in \mathcal{C}$ there is a countable neighbourood basis of 1 $\mathcal{N}_G$ consisting of open subgroups. Assume that $\{(G, U) \mid U \in \mathcal{N}_G\}$ is Borel and that for all $\varphi\colon G \cong H$ we have $U \in \mathcal{N}_G \implies \varphi(U) \in \mathcal{N}_A$.

Then isomorphism on $\mathcal{C}$ is Borel reducible to GI.

Apply this to:

1. $\mathcal{C}$ some class of locally compact groups, $\mathcal{N}_G$-compact open subgroups
2. $\mathcal{C}$ oligormophic, $\mathcal{N}_G$ all open subgroups

Proof Sketch. Find a language L such that $G \cong_{\mathrm{top}} H \iff M_G \cong_L M_H$. The universe of M_G is made of all the left and right cosets of subgroups in $\mathcal{N}_G$. We have predicates for left and right cosets, a binary relation interpreted as inclusion between cosets, and a ternary relation $A \cdot B \subseteq C$. Note that this allows to define a subgroup, by $A \cdot A \subseteq A$. For $g \in G$, consider $L_g = \{gH \mid H \in \mathcal{N}_G\}$ and $R_g = \{Hg \mid H \in \mathcal{N}_G\}$. L_g and R_g satisfy:

- L_g, R_g are downward directed with respect to inclusion
- For $A \in L_g$, $B \in R_g$ there is $C \in L_g$ such that $C \subseteq A \cap B$
- For each $H \in \mathcal{N}_G$ there is $gH \in L_g$ and HgR_g.

□

It is not clear if one can do better (i.e. see if the actual equivalence class for the problem for oligomorphic groups is lower than GI).

4.4 Alf Onshuus – Hrushovski's stabilizer theorem and groups in NTP2 theories

[joint with [Samaria?] and [someone]]

The paper is about groups definable in bounded PRC fields with strong f-generics. This will be in another talk in this conference.

We will talk of groups definable in a *geometric* NTP_2 *field* with *strong f-generics.*

Definition 4.61. Let k be a field (in the pure language of rings). It is *geometric* iff

- model-theoretic algebraicity is given by actual algebraicity, and
- it eliminates $\exists^\infty$.

Definition 4.62. $p(x)$ is *strong f-generic over* A iff for all $g \in G(\mathfrak{U})$ the translate $f \cdot p(x)$ does not fork over A. With *strong bi-generic* we meant that same happens both with left and right translates.

Example 4.63. $\mathbb{Q}_p$, RCF, pseudofinite fields.

Theorem 4.64 (Hrushovski, Pillay)**.** If G is definable in a geometric field k, then you can find $a, b, c \in G$ and $a', b', c' \in H$, an algebraic group definable over k, such that $a \cdot_G b = c$, $a' \cdot_H b' = c'$, $\operatorname{acl}(Aa) = \operatorname{acl}(a'A)$, $\operatorname{acl}(Ab) = \operatorname{acl}(b'A)$ for some A such that $\dim(a, b, c/A) = 2\dim(G)$, where the dimension is computed with acl.

If you have $a \cdot b = c$ you can try to get $a_1 \cdot b_1 = c$ in a way that ab and $a_1 b_1$ are independent in the transcendence degree sense. Then $(a_1^{-1} \cdot a)b = b_1$. In this case $(a_1^{-1} \cdot a)p(x) \cap p(x)$ is "large". We call the [set of such a?] $\operatorname{st}(p(x))$.

We try to look at $\operatorname{st}(b, b'/A) \subseteq G \times H$. Let $\operatorname{Stab}(p(x)) = \langle \operatorname{st}(p(x)) \rangle$.

In the examples above, people have been able to find isogenies between G and H.

We still have to say what "large" means.

- In F pseudofinite, taking the non-forking ideal gives a type-definable groups isogenic groups G' and H'. So here "large" is "not being in the forking ideal"
- With R real closed field definable sets and G, bounded do not extend to finitely satisfiable generics. [wut?]

[slides from now on]

Definition 4.65. Fix μ an A-invariant ideal on definable sets. Let p be a global type. Then $\operatorname{st}(p)$ is the set of all a's such that $ap(x) \cap p(x)$ is not in μ. Equivalently, a is in $\operatorname{st}(p)$ iff there is $b \models p \restriction A$ such that $\operatorname{tp}(b/Aa)$ is μ-large and $ab \models p$. Let $\operatorname{Stab}(p)$ be the group generated by $\operatorname{st}(p)$.

Definition 4.66. Let μ be an A-invariant ideal on defianble sets. Recall that μ has the S1 property if given any $\varphi(x, a_0)$ and $\varphi(x, a_1)$, for a_0 and a_1 starting an indiscernible sequence, if $\varphi(x, a_0) \wedge \varphi(x, a_1)$ is in μ then so is [what?]

Proposition 4.67. Let p be a type and assume that μ has the S1 property. Then for any type q the relation $R(a, b)$ defined as "$p(x)a^{-1} \cap q(x)b^{-1}$ is μ-wide[4]" where we identify a type with its realisationsin the monster model, is a stable relation.

[proof]

Theorem 4.68. Let μ be medium[5] in p, q. Then the relation "$gp \cap hq$ is wide" is a stable relation.

Fact 4.69. Let p, q be complete types over a model M and let $R(x, y)$ be a stable M-invariant relation in the realisations of $p(x) \times q(y)$. Then the truth value of [too fast]

Lemma 4.70. Assume that μ is S1 in p, r and $p^{-1}r$ and all of them are μ-wide; Let $(a, b) \models p \times_{\mathrm{nf}} p$. Then $ba^{-1} \in \mathrm{st}(r)$.

Theorem 4.71 (Hrushovski's stabiliser theorem (maybe))**.** Let μ be an M-invariant ideal on G. Let $p \in S_G(M)$ e wide. Assume that μ is S1 on $(pp^{-1})^2$ and invariant under (left and) right multiplication. Then $\mathrm{Stab}(p) = \mathrm{st}(p)^2 = (pp^{-1})^2$ is a connected, wide type-definable group on which μ is S1.

[proof idea]

Fact 4.72. In NTP_2 amenable groups, definable sets which don't extend to Bi-f-generic types form an S1 ideal μ_G.

We may assume a, b satisfy a bi-f-generic type $p(x)$. Pull back μ_G to an ideal μ on $G \times H$ defined to be sets that project in G to a set in μ_G. μ will be S1 on definable sets in $G \times H$ that have finite fivers in H.

Under NTP_2 if $\mathrm{Stab}(p)$ with $p := \mathrm{tp}(b, b'/M)$ is type-definable and μ-wide, then it would be a subset of $G \times H$ with finite projections, and must project into a set containing G^{00}.

[too fast; statement of the main theorem is quite long]

Corollary 4.73. Let G be a torsion free group[6] definable in a real closed field $\mathcal{R}$ (so $G = G^{00}$). Then there is an algebraic group H such that G is definably isomorphic to $H(\mathcal{R})$.

[4] I.e. not in μ.
[5] Missed what it means.
[6] This automatically gives definable amenability.

4.5 Özlem Beyarslan – Geometric Representation in Pseudo-finite Fields

[slides]

Theorem 4.74 (Ax)**.** If F is perfect, PAC and $\mathrm{Gal}(F) = \hat{\mathbb{Z}}$, then it is pseudo-finite.

Fact 4.75. If $(A, \sigma) \models \mathsf{ACFA}$, then $\mathrm{Fix}(\sigma)$ is pseudofinite.

Fact 4.76. $\mathrm{Fix}(\sigma)$ is pseudo-finite for almost σ in $\mathrm{Gal}(\mathbb{Q})$ (wrt the Haar measure).

Definition 4.77. Calla field F *bounded* iff it has finitely many extensions of degree n for each n. In this case the abosulte Galois group $\mathrm{Gal}(F)$ is called *small*.

The theory of a perfect PAC field F is determiend by its absolute Galois group, and the algebraic [something] [. . .]

Definition 4.78. A finite group G is *geometrically represented* in a theory T with elimination of imaginaries iff there are structures (A and B are subfields)$M_0 \leq A \leq B \leq M$ such that

- $M_0 \prec M \models T$
- $\mathrm{dcl}(A) = A$ [doesn't t always happen?]
- $B \subseteq \operatorname{acl} A$
- $\mathrm{Aut}(B/A) = G$

A prime is geometrically represented in T if p divides the order of some finite G which is geometrically represented.

Here $\mathrm{Aut}(B/A)$ is the set of permutations of B preserving the truth value (computed in M) of A-formulas.

If a finite group G is geometrically represented in a complete T over M_0, then it is represented over every elementary extension M of M_0.

Example 4.79. Let $T = \mathsf{ACF}_0$. Every finite group is represented. Since every finite group G is iso to a subgroup of the symmetric group acting on G. We have $\mathbb{C} \leq \mathbb{C}(x_1, \ldots, x_n)^G \leq G \leq \mathbb{C}(x_1, \ldots, x_n) \leq [\mathfrak{U}?]$.

Let p be prime. Let ζ_p be a primitive pth root of unity, μ_{p^n} the set of p^n-th root, μ_{p^∞} [that]. [. . .]

Theorem 4.80. Let F be a quasifinite[7] field, $\mathrm{char}(F) \neq p$, if p is geometrically represented in $\mathrm{Th}(F)$, then $\mu_{p^\infty} < F(\zeta_p)$.

[7] Perfect and Galois group $\hat{\mathbb{Z}}$

More precisely

Theorem 4.81. Let F be a quasifinite field, p prime. Assume p is geometrically represented in $\mathrm{Th}(F)$. Then

- If char $\neq p$ then...
- Otherwise it contains Ω [maximal p-extension or something like that]

Theorem 4.82. If a pseudo-finite F contains μ_{p^∞} then all $\mathbb{Z}/p^n\mathbb{Z}$ are represented.

Lemma 4.83. [...]

If F contains μ_{p^∞} for all primes p then every finite abelian group is represented.

Question 4.84. Which are the finite groups that can be represented?

Theorem 4.85 (Beyarslan, Chatzidakis)**.** If F is pseudofinite and A is a definably closed subfield, then $G/\operatorname{Aut}(\operatorname{acl}(A)/A)$ is abelian. Moreover, for any prime p dividing the cardinality of G, $p \neq \mathrm{char}(F)$, and $\mu_{p^\infty} \subseteq F$.

In a valued field, if v is Henselian, i.e. it has a unique extension to the separable closure, then $\mathrm{Gal}(K)$ is compatible with w, i.e. $w(\sigma(x)) = w(x)$ for every $x \in K^{\mathrm{sep}}$. This induces a canonical surjection π with

$$1 \to T \to \mathrm{Gal}(K) \xrightarrow{\pi} \mathrm{Gal}(k_\nu \to 1$$

where T is the inertia subgroup of $\mathrm{Gal}(K)$ wrt [...] [...]

Theorem 4.86 (Koenigsmann product theorem)**.** If K is a field with $\mathrm{Gal}(K) \cong G_1 \times G_2$, both non-trivial, let $\pi\colon \mathrm{Gal}(K) \to \mathrm{Gal}(K_\nu)$ be the canonical surjection[8]. Then $\mathrm{Gal}(k_v) = \pi(G_1) \times G_2)$ and the cardinalities of the $\pi(G_i)$ are coprime. If p divides the gcd of the cardinalities of the G_i, then $\mathrm{char}(K) \neq p$, $\mu_{p^\infty} \subseteq K(\zeta_p)$ and there is a non-trivial Henselian valuation on K.

Theorem 4.87 (Prestel)**.** Let K be a PAC field which is not separably closed. Then K has no Henselian valuation on it.

[proof of Theorem 4.85]

4.6 Gareth Boxall – The definable (p, q)-conjecture

[joint with C. Kestner]

This is about Conjecture 2.11 (that was stated in "externally definable sets and dependent pairs II" by Chernikov and Simon).

[8] v is the canonical Henselian valuation.

Theorem 4.88 (Chernikov, Simon)**.** Using the (p,q)-theorem one gets $\psi(y) \in \mathrm{tp}(b/M)$ such that $\psi(M)$ is covered by finitely many sets of the form $\varphi(a,M)$.

Theorem 4.89 (Boxall, Kestner)**.** Conjecture 2.11 holds for distal theories.

Also, there is a different proof of

Theorem 4.90. If $\mathrm{Th}(M^{\mathrm{ext}})$ is distal, then $\mathrm{Th}(M)$ is distal.

[definition of distality (NIP build in the definition)] [Theorem 2.8] [A Lemma explained in pictures; not sure if in Lotte's talk]

To prove Conjecture 2.11 for distal theories, choose $\psi(y) \in \mathrm{tp}(b/M)$ well- For each n let $\varphi_n(x,\bar{y})$ be $\bigwedge_{i\leq n}\varphi(x,y_i)$ and let $f(n)$ be the number of sets $\varphi_n(a,M)$ needed to cover $\psi(M)^n$. If $f(1)>1$, then $f(n)\to\infty$ by a combinatorial argument. The idea is use strong honest definitions to get a contradictions by controlling something that's growing with something that's not growing.

This Lemma came up in the last few days to prove that Theorem about M^{ext}.

Lemma 4.91. Let T be NIP in L. Let $M \models T$. Let $M^{\mathrm{ext}} \prec^{+} N$. Let $A \subseteq N$ be an L-indiscernible sequence. Then A extends to an L-indiscernible sequence A' such tat if $q(\bar{x})$ is a complete L-n-type over A' which is finitely realised in A', then $q(\bar{x})$ implies a complete type over A' in L^{ext}.

Chapter 5

Tuesday 4 July

5.1 Sergei Starchenko – Model Theory and Combinatorial Geometry, I

[slides] [joint with Chernikov and Galvin]

Fix $\mathcal{F} \subseteq \mathscr{P}(X)$. Look at the incidence relation by considering a bipartite graph $\mathcal{G}_{\mathbb{I}} \coloneqq (x, \mathcal{D}, \mathbb{I})$. We study the family of finite subgraphs of this.

[example]

Assume $\mathbb{I}$ is definable in a first order structure $\mathcal{M}$. What are the combinatorial properties of $\mathcal{G}_{\mathbb{I}}$ assuming that $\mathcal{M}$ (global) of $\mathbb{I}$ (local) is NIP, stable, ecc?

[example]

Definition 5.1. A relation $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ has the *strong Erdös-Hajnal property* iff there is $\delta > 0$ suc that for any $(U, V, I \in \mathcal{G}_{\mathbb{I}}$ there are $U_0 \subseteq U$, $V_0 \subseteq V$ of size $\geq \delta$ and either $(U_0 \times V_0) \cap I = \emptyset$ or $(U_0 \times V_0) \subseteq I$.

Theorem 5.2 (Chernikov-Starchenko)**.** If a relation is definable in a distal structure then $\mathcal{G}_{\mathbb{I}}$ has the Stong Erdös-Hajnal Property.

Example 5.3. If $\mathbb{F} \vDash \mathsf{ACF}_p$ then if $I \subseteq \mathbb{F}^2 \times \mathbb{F}^2$ is the set of pairs (u, v) with $u_1 v_1 = u_2 + u_2$, then this does not hat the Strong Erdös-Hajnal Property.

Definition 5.4. $\mathbb{I}$ is NIP if there is $d \in \ mbN$ such that for all finite $B \subseteq \mathbb{V}$ we have [too fast]

[too fast]

[there is a local notion of distality]

Theorem 5.5. If $\mathbb{I}$ is distal in some NIP structure $\mathcal{M}$, then $\mathcal{G}_{\mathbb{I}}$ has the Strong Erdös-Hajnal Property.

[one of the main tools is called *cutting lemma*; it was proved some years ago; now they have an *optimal cutting lemma*]

5.2 Artem Chernikov – Model Theory and Combinatorial Geometry, II

[slides] Sergei has presented one technical result that started this investigation. This talk is about where the program is going.

Let $G = (U, V, I)$ with $I \subseteq U \times V$ be a bipartite graph, with U, V infinite. For $A \subseteq U$, $B \subseteq V$, $I(A, B)$ is the induced subgraph.

Let $K_{k,k}$ be the complete bipartite graph with each part of size k (graphs will be $K_{k,k}$-free).

Fact 5.6 (Kovari, SOs, Turan 1954). For each $k \in \mathbb{N}$ there is some $c \in \mathbb{R}$ such that: for any bipartite graph G and $A \subseteq U$, $B \subseteq V$ with $|A| = |B| = n$, if $I(A, B)$ is $K_{k,k}$-free, then $|I(A, B)| \leq cn^{2-\frac{1}{k}}$

For simplicity, we only discuss the balanced case, i.e. $|A| = |B| = n$. Most of the results have unbalanced versions as well.

Theorem 5.7 (Bohman, Keevash, 2010). For all $k \geq 5$, there is a bipartite $K_{k,k}$-free graph with $\geq cn?2 - \frac{2}{k+1}$ edges.

Let U be an infinite set and $\mathcal{F}$ a family of subsets of U. For $A \subseteq U$ let $\mathcal{F} \cap A = \{S \cap A \mid S \in \mathcal{F}\}$. Let $\pi_{\mathcal{F}}(n)$ be the shatter function $\max\{|\mathcal{F} \cap A| mi\ A \subseteq U, |A| = n\}$.

Definition 5.8. The VC-density of $\mathcal{F}$ is $\inf\{(r \in \mathbb{R} \mid \pi_{\mathcal{F}}(n) = O(n^r)\}$, or ∞ if there is no such r.

Given $I \subseteq U \times V$, we have the family $\mathcal{F}_I = \{I_b \mid b \in V\}$ of subsets of U, where $I_b := \{a \in U \mid (a, b) \in I\}$. Let $\mathrm{VC}(I) := \mathrm{VC}(\mathcal{F}_I)$.

Example 5.9 (Sauer-Shelah Lemma). If $\mathcal{M}$ is NIP and I is definable, it has finite VC-density.

Example 5.10 (Aschenbrenner, Dolch, Haskell, Machperson, Starchenko). [...]

Fact 5.11 (Fox, Pach, Sheffer, Suk, Zahl, 2015). For every d, k there is $c \in \mathbb{R}$ such that the folllowing holds. Let G be a bipartite graph with $\mathrm{VC}(I) \leq d$. THen for any $A \subseteq U$, $B \subseteq V$ with $|A| = |B| = n$, if $I(A, B)$ is $K_{k,k}$-free, then $|I(A, B)| \leq cn^{2-\frac{1}{d}}$.

Conversely, independence of the bounding exponent [constant?] from k implies that I is NIP. In particular, if $d = 2$, the exponent is $3/2$. This case is particularly important (we will see later).

In $\mathcal{K} \models \mathsf{ACF}_p$ we have a matching lower bound:

Example 5.12. Let $U = V = K^2$, (squares) let $\mathbb{F}_q \subseteq K$ be a finite field, and let $A = (\mathbb{F}_q)^2$ be the set of points on the plane over $\mathbb{F}_q$. Let B be the set of all lines, and I be the (definable) incidence relation. Then $\text{VC}(I) = 2$ and I is $K_{2,2}$- free (only one line between two points). We have $|A| = |B| = q^2$ and [. . .]

Over the reals a better bound holds (optimal up to a constant by Erdös]

Fact 5.13 (Szemeredi-Trotter 1983)**.** Let I be the incidence relation between points and lines on the affine plane over $\mathbb{R}$. Then $|I(A, B)| = O(n^{\frac{4}{3}})$.

Note that $\frac{4}{3} < \frac{3}{2}$. In fact, even in ACF_0 the bound is better.

Fact 5.14 (Toth '03)**.** In $\mathbb{C}$ it is 4/3

The reason is that ACF_0 is a reduct of a distal theory, while ACF_p is not. More precisely, because the cutting lemma (previous talk) holds in ACF_0.

This is a generalisation of a result by Fox, Pach, Sheffer, Suzk, Zahl in the semialgebraic case

Theorem 5.15. Let $\mathcal{M}$ be an o-minimal expansion of a field, $I(x, y) \subseteq M^2 \times M^2$ definable. Then for any $k \in \omega$ there is c such that: for any $A, B \subseteq M^2$, if $I(A, B)$ is $K_{k,k}$-free, then $|I(A, B)| \leq cn^{\frac{4}{3}}$.

Basue and Raz got independently the same conclusion under a stronger assumption that the whole I is $K_{k,k}$-free. Their proof uses the crossing number inequality, that appears specific to o-minimality.

Theorem 5.16 (Distal cutting lemma)**.** Assume $I(x, y)$ admits a distal cell decomposition $\mathcal{T}$ with $|\mathcal{T}(S)| = O(|S|^d)$. Then there is a constant c such that for any finite $S \subseteq M^{|y|}$ of size n and any real $1 < r < n$ there is a covering $X_1, \ldots, X_t$ of $M^{|x|}$ with $t \leq cr^d$ and each X_i crossed by at most n/r of the sets $\{I(x, b) \mid b \in S\}$.

Theorem 5.17 (Optimal distal cell decomposition)**.** If $\mathcal{M}$ is an o-minimal expansion of a field and $I(x, y)$ with $|x| = 2$ definable. Then $I(x, y)$ admits a distal cell decomposition $\mathcal{T}$ with $|\mathcal{T}(S)| = O(|S|^2)$ for all fintie sets S.

Combining, every $I \subseteq M^2 \times M^2$ has an r-cutting of quadratic size, and $\text{VC}(I^*) = 2$ by o-minimality. Starting with the general incidence bound given by the VC-density in o-minimal structures, recursively can improve it using cutting lemma for a certain careful choice of r.

Even without precise bounds, one can get

Theorem 5.18. Let $\mathcal{M}$ be strongly minimal, interpretable in a distal structure, and $I \subseteq M^2 \times M^2$ is $K_{k,2}$-free. Then there is some $\varepsilon > 0$ such that if I is $K_{k,2}$-free, then $|I(A, B)| \leq n^{\frac{3}{2}-\varepsilon}$. ($\varepsilon$ depends on k)

This can be combined with the group configration theorem (Tao, rushovski, Raz-Scharir-Solymosi] to generalise Elekes-Ronyai to strongly minimal theories interpretable in distal theories.

Given a bipartite graph $G = (U, V, I)$, let $f(n) = \max\{|I(A, B)| \mid A \subseteq U, B \subseteq V, |A| = |B| = n\}$.

Definition 5.19. The upper density of G is $\bar{d}(I) = \inf\{c \in \mathbb{R} \mid f(n) = O(n^c)\}$.

Remark 5.20. $\bar{d}(I) \in \{0\} \cup [1, 2]$

Theorem 5.21 (Blei, Korner)**.** For any $\alpha \in [1, 2]$ there is some bipartite graph with $\bar{d}(I) = \alpha$.

What values can this take when I is definable in a nice structure? E.g.

Question 5.22. Can $\bar{d}(I)$ be irrational for I definable in a NIP structure?

Theorem 5.23 (Bukh, Conlon)**.** ($\approx$; not exactly what is in the paper, but should work) If $\mathcal{K}$ is a pseudofinite field, then $\bar{d}(I)$ can be any rational $\alpha \in [1, 2]$

If I is the point-line incidence relation on the affine plane over a field K, then $\bar{d}(I)$ is $4/3$ if $\mathrm{char}(K) = 0$, and $3/2$ in characteristic p.

Theorem 5.24. Assume that $\mathcal{M}$ is o-minimal and $I \subseteq M^2 \times M^k$ is a definable rlation with $\bar{d}(I) \in (1, 2)$. Then $\mathcal{M}$ defines a field.

The reason is that there are strong bounds int he locally modular case, and also the trichotomy in o-minimal structures.

Definition 5.25. $\mathcal{M}$ is combinatorially linear iff for every definable $I(x, uy)$, $\bar{d}(I) \in \{0, 1, 2\}$.

By the remark above, such a structure cannot define a field.

Definition 5.26. A formula $I(x, y)$ is weakly normal if there is k in $\mathbb{N}$ such that the intersection of any k pairwise distinct sets of the form I_b, $b \in M^{|y|}$ is empty.

Definition 5.27. T is 1-based if every formula is a Boolean combination of weakly normal formulas.

This implies stability, and is equivalent to the definition in terms of forking. Stable 1-based theories satisfy a linear Zarankiewicz bound.

Theorem 5.28. Let $\mathcal{M}$ be stable, 1-based. Then for every defianble $I(x, y) \subseteq M^{|x|} \times M^{|y|}$ and $k \in \mathbb{N}$ there is some $c \in \mathbb{R}$ satisfying: for any finite A, B, if $I(A, B)$ is $K_{k,k}$-free, then $|I(A, B)||lec(|A| + |B|)$.

In particular, this implies that $\mathcal{M}$ is combinatorially linear.

Conjecture 5.29. For any definable $I(x_1, \dots, x_k)$, $\bar{d}(I) \in \mathbb{N}$.

Question 5.30. Characterise combinatorial linearlity ammong stable, ovr even strongly minimal, structures (Hrushovski constructions?)

Conjecture 5.31. Let $\mathcal{M}$ be o-minimal, locally modular. Then every definable $I(x, y)$ satisfies a linear Zarankiewicz bound.

It seems defficult even for $I \subseteq \mathbb{R}^2 \times \mathbb{R}^4$ the incidence relation between points and rectangles on the plane.

Fact 5.32 (Discussion with Sheffer)**.** For 2-parametric families on the plane, $|I(A, B)| \leq c(n \log n)$.

Theorem 5.33. Let $\mathcal{M}$ be o-minimal, locally modular. If $I \subseteq M^2 \times M^d$ is definable an $I(M^2, M^d)$ is $K_{k,k}$-free, then $|I(A, B)| \leq c(|A| + |B|)$.

[proof idea] [unit distance problem gets in the way]

5.3 Itaï Ben-Yaacov – A talk which has nothing (much) to do with Amalgamation in Globally Valued Field

[slides]

The sum formula is something like: $\forall x \in K^\times \ \sum_\omega m_\omega \cdot v_\omega(x) = 0$. Or more generally $\int_\Omega v_\omega(x) \, \mathrm{d}\mu(\omega) = 0$.

Example 5.34. $K = \mathbb{Q}$ and Ω consistes of all v_p for prime p together with the archimedian valuation $v_\infty(x) = -\log|x|$ (the last one is archimedean.

Example 5.35. $K = k(t)$, Ω is all v_p for prime $p \in k[t]$ together with $v_\infty = -\deg f$.

[...]

Definition 5.36. A globally valued field is a field K together with a measured family of valuations $v_\omega\colon \ \to \mathbb{R} \cup \{\infty\}$ such that $\int v_\omega(x) = 0$ for all $x \neq 0$. (so valuations should have real values)

Remark 5.37. This is an inductive elementary class (first order, $\forall\exists$ axioms) in continuous logic (and even slightly worse) in *continuous logic* (and even slightly worse).

The motivation in studying this class is: render certain questions regarding number and functions fields, heights,..., amenable to model theory. Also, it is difficult and makes you do interesting math.

Since the class of globally valued fields is elementary, it has a theory GVF.

Question 5.38.

- Does GVF has a model companion?
- Is the class of e.b. modls elementary? How does one axiomatise "richness" in GVFs=

Question 5.39. Is the model companion a model completion (does it have QE)? No: amalgamation over an arbitrary GVF

Question 5.40. Is the model companion stable? There is no obvious obstruction: the many valuations are coded in an L^1 Banach spaces which is stable.

There are two approaches: Hrushovski and Ben-Yaacov (missed the details).

Conjecture 5.41. The model companion GVF^* is just GVF plus the *fullness axiom*: $\forall v_\omega\ \exists x \ldots \to \exists x \forall v_\omega \ldots$.

This holds in every existentially closed GVF, but does it suffice?

Conjecture 5.42. (a certain estimate holds and it is interesting)

Theorem 5.43 (Spring 2016, assuming the estimate)**.** Let K be a GVF (non-trivial, alg. closed, surjective...). Then K is an amalgamation base ("weakly e.c.") if and only if K is full.

Theorem 5.44 (Autumn 2016)**.** The estimate holds.

Let's play an analogy game. Choose A, solve for X.

- $\{T, F\}$ is to $\mathbb{R}$ as A is to X

E.g.

- Classical logic →continuous logic
- $\forall, \exists \to \sup, \inf$
- Equality → distance
- Set → metric space (complete)
- Topological spaces → topometric spaces
- Compactness → comapctness
- Stability → stability
- Algebraic geometry → ??? Rigid analytic geometry? More algebraic?

So let's see what you look at in algebraic geometry

- Field→ complete metric field: $\mathbb{R}$, $\mathbb{C}$ or (K, v)
- Ring → complete metric ring?
- Ring $= K[X]/I \to K[X]$modulo…something?

Definition 5.45. A *sub-valuation* on a ring A is a function $A \to \mathbb{R} \cup \infty$ such that

- $u(ab) \geq u(a) + u(b)$ (if = holds, then ti is a valuation)
- $u(a^n) = nu(a)$
- $u(a + b) \geq \min u(a), u(b)$
- $u(0) = \infty$

Idea: a vlauation (sub-valuation) $u : A \to \{0, \infty\} = \{T, F\}$ is just a prime (radical) ideal.

So in the analogies before,

- $K[X]/I \mapsto K[X]$ with a sub-valuation
- Integral domain $\mapsto K[X]$ with a valuation

Theorem 5.46 (Ben-Yacoov, 2014)**.** Like ACF, the theory ACMVF of algebraically closed metric valued fields, in continuous logic, has QE and is stable.

Definition 5.47. Say K is a valued field. A valuation on a K-vector space E is a function $u_E : E \to \mathbb{R} \cup \{i\}nfty$ such that

- $e_E(ax) \geq v_K(a) + e_E(x)$ (actually = holds)
- $u_E(a + b) \geq \min\{u_E(a), u_E(b)\}$

The *Tensor product valuation* on $E \otimes_K F$ is the least such that $(u_E \otimes u_F)(x \otimes Y)$ is the sum

If $K \subseteq L_M$ are ACF, then $L \otimes_K M$ is an integral domain.

Theorem 5.48 (Poineau 2013, Ben-Yacoov 2015)**.** […]

Round three of the analogy game:

- Radical ideal → sub-valuation
- Zariski closed → …?

Definition 5.49. Let u be a [homegenous, almost finitely generated[1]] subvaluation on $K[X] = K[X_0, \ldots, X_n]$

$$\operatorname{Ker} u = \{P \mid u(P) = \infty\} \qquad W = V(\operatorname{Ker} u) \subseteq \mathbb{P}^n$$

$$u^*(\xi) = \inf_{\text{homog.}P} v(P(x)) - u(P) - \hat{v}(x)$$

where $\xi = [x] \in W$ and $\hat{v}(x) = \min_i v(x_i)$.

Such a function $\eta = u^* \colon W \to \mathbb{R}$ is a *virtual divisor* on W.

So the analogue of a Zariski closed W is a (W, η) with η a *virtual divisor*. And actually

Theorem 5.50 ("Valued Nullstellensatz")**.** Bijection $u \leftrightarrow (W, u^*)$. Converse duality $\eta = u^* \iff u = n^*$.

Lemma 5.51. Let $(L, v) \supseteq (K, v)$. Let $f \in L[X]_m$ (homog. deg m) have no zeros on $\mathbb{P}(K)$. Then $\eta(\xi) = \hat{f}(\xi) = v(f(x)) - \hat{v}(x)$ ($\xi = [x] \in \mathbb{P}^n(K)$) is a virtual divisor, and every virtual divosr is a uniform limit of such.

In other words, a virtual divosr is the "echo" of an actual divosr from a larger model. In model theoretic terms, it is an externally definable ($\mathbb{R}$-valued) predicate. By stability, a virtual divosr is a definable predicate (so, no need to use externals).

Fact 5.52 (Resultant)**.** There is an irreducible polynomial $\mathfrak{C}$ over $\mathbb{Z}$ which takes the coeffients of $n+1$ homogenous polynomials in $n+1$ unknowns, and vanishes if and only if they have a common zero in $\mathbb{P}^n$. Let us write $f_0 \wedge \ldots \wedge f_n = \mathfrak{C}(f_0, \ldots, f_n)$.

Proposition 5.53. For $i = 0, \ldots, n$ assume that

- L_i/K is an extension
- $f_i \in L_i[X]_{d_i}$
- $\eta_i = \hat{f}_i$ is a virtual divosr.

Then in the free amalgam $\operatorname{Frac}(L_0 \otimes_K \ldots \otimes_K L_n)$

$$\eta_0 \wedge \ldots \eta_n := \frac{v(f_0 \wedge \ldots, f_n)}{d_0 \ldots d_n} \in \mathbb{R}$$

depends only on the η_i, i.e. the traces of the f_i on K

Summing up: Zariski-closed $W \subseteq \mathbb{P}^n \to$ a pair (W, η). η is a virtual divisor, i.e. the trace on W of a poluunomial with external coefficients. Assume for simplicity that $W?\mathbb{P}^n$. Then to η we associate $\eta^{\wedge n+1} = v(\ldots)/d^{n+1} \in \mathbb{R}$, where $\ldots$ is the resultant of independent copies of f. [...]

[1]Needs to be assumed, there is no Hilbert basis theorem, and that is why it is not superstable.

Definition 5.54 ("volume=determinant"). Let E be a valued K-vector space, $x = (x_0, \dots, x_{k-1})$ a basis. $x^\wedge = \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot x_{\sigma(0)} \otimes \dots, \otimes x_{\sigma(k-1)} \in E^{\otimes k}$. Call $\mathrm{vol}_x(E, u_E) = u_E^{\otimes k}(x^\wedge)$. [depends on a choice of basis]

Theorem 5.55. Let η be a virtual divisor on $\mathbb{P}^n$ and η^* [...] [estimate mentioned before]

[proof sketch]

Locally, [that estimate holds]. GLobally, let K be a GVF. At each valuation v_ω, let η_ω be a virtual divisor and η_ω^* the dual sub-valuation on $K[X]$. Integrating both sides (basically same estimate), because since K is a GVF, volume is independent of the basis. This generalises [the estimate is for $\mathbb{P}^n$, there is an estimate also for any W]. [...]

Definition 5.56. Say that a GVF K is full iff it is (non-trivial, algebraically closed, surjective) and for every valued vecotr space (E, u)

$$\forall \varepsilon > 0 \, \exists x \in E \setminus \{0\} \int u(x) > \frac{\mathrm{vol}(E, u)}{\dim_E} - \varepsilon$$

Corollary 5.57. With the earlier hypothesis, there is $W' \subseteq W$ of dimension $\ell - 1$ such that[estimate].

Corollary 5.58 (Full GVFs are linearly e.c.). Let K be a full GVF, $E = (E, u)$ a valued vector space. Let $L \supseteq K$ be a larger GVF, $E_L = (E \otimes_K L, u \otimes v_L)$. If there exists $x \in E_L$ suc that $\int u_L(x) > 0$ then such x already exists in E.

Corollary 5.59. Let K be a full GVF L_1 and L_2 two GVF extensions. Then both embed over K in some larger GVF M.

[proof]

5.4 Dugald Macpherson – Cardinalities of definable sets in finite structures

[slides] [joint with Anscombe, Steinhorn, Wolf]

Theorem 5.60 (Chatzidakis, van den Dries, Macintyre, 1992). Let $\varphi(x_1, \dots, x_n; y_1, \dots, y_m)$ be a formula in the language of rings. There is a positive C and finitely many pairs (d_i, μ_i) $(1 \le i \le K)$ with $d_i \in \{0, \dots, n\}$ and $\mu_i \in \mathbb{Q}^{>0}$ such that for each finite field $\mathbb{F}_q$ and $\bar{a} \in \mathbb{F}_q^m$ if the set $\varphi(\mathbb{F}_q^n, \bar{a})$ is nonempty, then $||\varphi(\mathbb{F}_q^n, \bar{a})|| - \mu_i q^{d_i} < C[\dots]$ [more stuff]

The CDM theorem was turned into a definition of an abstract model-theoretic framework in work of Macpherson and Seinhorn, Elwes, Rythen,... [1-dimensional asymptotic classes]

Elwes (2007): notion of N-dimensional asyptotic class.

Ryten (PhD thesis 2007): for any fixed Lie type τ, the class of all finite simple groups of type τ is an asymptotic class. The point is that these groups are very close to the corresponding finite field, so there is a uniform interpratation.

There is a corresponding notion of *measurable* infinite structure (Machperson, Steinhorn) (implies supersimple, finite SU rank) e.g. pseudofinite fields. But gives much more, for instance an ACF_0 is not measurable.

Fact 5.61. An ultraproduct of any asymptotic class is measurable.

We aim to broaden this framework, e.g. allow parts of the structure (sorts? coordinatising geometries?) to vary independently, not require that ultraproducts hav finite rank, or even have simple theory, not be specific about the form of the functions giving approximate cardinalities, i.e. not just of the form μq^d.

POssible examples to keep in mind are:

- Pairs $(V, \mathbb{F}_q)$ (2-sorted language), V a finite-dimensional vector space over $\mathbb{F}_q$-
- Disjoint unions of complete graphs all of the same size (n copies of K_m)
- Finite abelian groups
- Finite graphs of bounded degree

For a class $\mathcal{C}$ of finite L-structures and a tuple y of fvariables, let $(\mathcal{C}, y)$ be the set $\{(M, a) \mid M \in \mathbb{C}, a \in M^{|y|}\}$ of pairs consisting of a structure in $\mathcal{C}$ and a y-tuple from that structure ("pointed structures in $\mathcal{C}$").

A finite partition Φ of $(\mathcal{C}, y)$ is $\emptyset$-definable iff[...][missing]

Definition 5.62 (R-multidimensional asymptotic class)**.** Let R be a set of functions $\mathcal{C} \to \mathbb{R}_{\geq 0}$. A class $\mathcal{C}$ of finite L-structures is an R-multidimensional asymptotic class (R-mac) if for every $\varphi(x, y)$ there is a finite $\emptyset$-definable partition Φ of $(\mathcal{C}, y)$ and a set $H_\Phi \coloneqq \{h_p \in R \mid P \in \Phi\} \subseteq R$ such that for each $P \in \Phi$

$$||\varphi(x; b)| - h_P(M)| = o(h_P(M)) \tag{5.1}$$

for $(M, b) \in P$ as $|M| \to \infty$.

An R-mec (multidimensional exact class) if above we have $|\varphi(x, b)| = h_P(M)$. Weak R-mac or R-mec are given by dropping the definability clause on the partition Φ.

Remark 5.63. To prove that a class $\mathcal{C}$ is an R-mac or R-mec it suffices to work with formulas $\varphi(x, y)$ where $|x| = 1$, replacing R by the ring generated by R (fibering argument, using definability, similar to cell decomposition in o-minimality).

Remark 5.64 (Wolf)**.** If $\mathcal{C}$ is a mac or mec, then so is any class of finite structures uniformly bi-interpretable with $\mathcal{C}$. (but these conditions are not closed under uniform interpretability, as the defianbility clause be may be lost; e.g. take a reduct).

Remark 5.65. Any class uniformly interpretable in a mac is a weak mac.

Example 5.66 (Garcia, Machpherson, Steinhorn)**.** Class $\mathcal{C}$ of 2-sorted structures $(V, \mathbb{F}_q)$ with V finite dim. v.s. over $\mathbb{F}_q$. Given $\varphi(x, y)$ there is a finite set E_φ of polynomials $g(\mathbb{V}, \mathbb{F})$ over $\mathbb{Q}$ such that if $M = (V, F)$ then each $h_P(M)$ has form $g(|V|, |F|)$ for some $g \in E_\varphi$. Ultraproducts of $\mathcal{C}$ are supersimple, but the V-sort may have rank ω.

More generally, fix a quiver Q (digraph) of *finite representation type* (A_n, D_n, E_q, E_7, E_8). Over a field F; this has a finite-dimensional path algebra FQ which has finitely many iso types of indecomposable representations. Let $\mathcal{C}_q := \{(V, FQ, F) \mid F\text{finite field,V,}\mathsf{finite\ module\ for\ }FQ\}$ (3-sorted, with the natural language). Then $\mathcal{C}_Q$ is an R-mac with the functions h_P given by polynomials $g(F, W_1, \dots, W_t)$, were the W_i variables correspond to the indecomposables.

Example 5.67 (Bello Aguirre)**.** In the language of rings, for fixed $d \in \mathbb{N}$, let $\mathcal{C}_d$ be the collection of all finite residue rings $\mathbb{Z}/n\mathbb{Z}$, where n is a product of powers of at most d primes, each wit exponent at most d. Then $\mathcal{C}_d$ is a weak mac, and a mac after appropriate expansion by unary predicates. If just one prime is involved, this is an asymptotic class, e.g. $\{\mathbb{Z}/p^2\mathbb{Z} \mid p \text{ prime}\}$ is a 2-dimensional asymptotic class. Utlraproducts are supersimple of fnite SU-rank (Idea: $\mathbb{Z}/p^d\mathbb{Z}$ is coordinatised by [...])

(Work with Harrison-Shermoen) Notion of an embedded mac (cf. "embedded finite model theory"): class $\mathcal{C}$ of structures of the form (M, N) were N is a finite substructure of the possibly infinite structure M. THe formula $\varphi(x, y)$ is interpreted in (M, N) with y ranging over M, but we only consider cardinalities of defianble sets in N or its powers. TO sow that $\mathcal{C}$ is an embedded mac it suffices to sow that the corresponding class of inite structures N is a mac and that in all ultraproducts the N-part is fulle (o.e. stably, canonically) embedded.

Example 5.68. THe class of structures $(\widetilde{\mathbb{F}_p}, \mathbb{F}_{p^n})$.

Example 5.69. The calss of all 2-sorted valued fields $(\mathbb{Q}_p, \mathbb{F}_p)$

Example 5.70. Given an infinite graph G with vertices of finite degree at most d, the calss of all structures (G, A), A a finite subset of G.

Example 5.71. For any ω-categorical ω-stable structure M, the calss of structures (M, N) where N is a finite envelope.

There is a correspodning generalisation of measurable structure. Let $(S, +, \cdot, 0, 1, , <)$ be a commutative ordered semiring. Define $\sim$ on S with $a \sim b$ iff $a \leq b \leq na$ or $b \leq a \leq nb$ for some $n \in \mathbb{N}$. Put $D := S/\sim$, and $d\colon S \to D$ the natural "dimension" map. Say S is a measuring semiring if

$$\forall x, y, z \in S(x < y \wedge d(y) = d(z)) \to x + z < y + z$$

[my question: does this not come free from the definition of ordere semiring?] Let S be a measuring semiring, and let M be an L-structure. We say that M is S-measurable iff there is a function $h\colon \mathrm{Def}(M) \to S$ such that

1. h is the cardinality for finite X
2. h is finitely additive
3. mac condition: for each $\emptyset$-definable family X there is a finite $F \subseteq S$ such that $h(X) = F$ and for each $f \in F$ the fiber is definable
4. A "Fubini" condition

Weakly generalised measurable is obtained by removing the definability condition.

Proposition 5.72. Let M be weakly generalisd measurable. Then

- M does not have SOP (the collection of finite interval cannot form an asymptotic class)
- M is *functionally unimodular*, that is if $f_i\colon A \to B$ for $i \in \{1, 2\}$ are definable surjections with f_i k_i-to-1 then $k_1 = k_2$. (note how this holds in finite structures)

Proposition 5.73. Let M be S-measurable, and let $S_0 = \{h(X) mid X \subseteq M \text{ definable}\}$. If $d(S_0) = S_0/\sim$ is well-ordered then M is supersimple (Idea: forking ensures drop in dimension).

Example 5.74 (Anscombe)**.** If M is a Fraïssé limit of a free amalgamation class then M is generalised measurable (note for example the generic triangle-free graph is such a Fraïssé limit hand has TP_1 and TP_2 theory.

Proposition 5.75. If $\mathcal{C}$ is a mac then any ultraproduct is generalised measurable (so NSOP ecc).

Proposition 5.76. If $\mathcal{C}$ is a mec then any ultraproduct is S-measurable for some ordered *ring* S.

Remark 5.77. The above supersimplcity result applies to ultraproducts of examples like $\{(V, \mathbb{F}_q) \mid q \text{ prime power}, V \text{f.d. over} \mathbb{F}_q\}$.

Examples of mecs:

Example 5.78 (Essentially by Pillay)**.** Let M be any psudofinite strongly minimal set. Then there is a mec whose infinite ultraproducts are all elementarily equivalent to M, with the functions determining cardinalitites given as polunomiasl (over $\mathbb{Z}$) in the cardinalities of the finite structures

Example 5.79 (Wolf, based on Cherlin, Hrushovski)**.** For a fixed L and $d \in \mathbb{N}$, let $\mathcal{C}_{L,d}$ be the collection of all finite L-structures with at most d 4-types. Then $\mathcal{D}_{L,d}$ is a mec ()functions determining cardinalities are given by polynomials in the coordinatising Lie geometries.

Example 5.80. For any fixed d; the class $\mathcal{C}_d$ of finite graphs of degree at most d is a mec.

Remark 5.81. Finite fields are not even a *weak* mec.

Example 5.82 (Help from Kestner)**.** The class of all finite abelian groups is a mec. In fact, for any fixed finite ring R; this holds for the class of all finite R-modules.

Proposition 5.83. If $\mathcal{C}$ is a mec of groups, ten there is $d \in \mathbb{N}$ such that the groups in $\mathcal{C}$ have (uniformly definable) soluble radical $R(G)$ of index at most d, and $R(G)/F(G)$ has derived length at most d (here $F(G)$ is the largest nilpotent normal subgroup of G).

Question 5.84. Is there a mec with an ultraproduct with non-simple theory?

Conjecture 5.85. If M is a homogenous structure over a finite relational language, then the following are equivalent:

1. There is a mec with ultraproduct elementarily equivalent to M
2. M is stable

Remark 5.86. $2 \Rightarrow 1$ follows from work of Lachlan and Wolf.

Remark 5.87. The Paley graphs form a mac (but not mec) with limit the random graph, which is unstable.

Remark 5.88. The conjecture holds for homogeneous graphs, by the Lachlan-Woodrow classification and the follwoing

Theorem 5.89. Let M be any of the following omogeneous structures. Then there is no mec with an ultraproduct elementarily equivalent to M.

1. any unstable homogenous graph
2. [random tournament?]
3. [more examples]

Proof. To see that the generic torunament is not a limit of a mec. For a contradcition, consider a mec of finite tournaments with ultraproduct $\equiv$ the random tournament. Any finite regular torunament has indegree equal to outdegree, so has an odd number of vertices. For any formula $\varphi(x, y)$, in a large enough finite torunament M the cardinality $|\varphi(M, a)|$ depends just on the iso type of a (uses QE and definability clause of mec). If M is large enough finite and a, b are distinct vertices, then the tournaments M and on the sets $\{x \mid a, b, \rightarrow x\}$, $\{x \mid x \rightarrow a, b\}$, $\{x \mid a \rightarrow x \rightarrow b\}$ and $\{x \mid b \rightarrow x \rightarrow a\}$ are all regular, so all of odd size. But then the sum of four odd numbers +2 is even! □

5.5 Samaria Montenegro – Definable groups in PRC fields

[slides] [joint with Onshuus and Simon]

Definition 5.90. A PAC field is a field M such that M is existentially closed in the language of rings into each *reglular* field extension of M.

Definition 5.91. Let M and N be fields of characteristic 0 such that $M \subseteq N$. We say that N is a regular extension of M if $n \cap M^{\mathrm{alg}} = M$.

Example 5.92. Algebraically closed or pseudofinite fields.

They cannot be ordered

Definition 5.93. A field M of characteristic 0 is PRC fi M is existentially clsoed wrt the language of rings into each regular field extension N to which all orderings of M extend.

Fact 5.94 (Prestel)**.** The class of PRC fields is axiomatisable in the language of rings.

Example 5.95. Examples:

- PAC fields of characteristic 0
- RCF
- $\mathbb{Q}_{\mathrm{tr}}$ the maxiaml totally real extension of $\mathbb{Q}$
- (another one)

Definition 5.96. A field M is called bounded iff for any integer n, M has only finitely many extensions of degree n.

Fix a bounded PRC field K and $K_0 \prec K$. The language is $L_{\mathrm{ring}} \cup \{c_m \mid m \in K_0\}$.

Fact 5.97. There is n such that K has only n orders.

Let $\{<_1, \dots, <_n\}$.

Remark 5.98. Each $<_i$ is definable in an existential L-formula (you need the constants)

[...][definition of T_{prc}

Theorem 5.99. Let $M \vDash T_{\text{prc}}$. Let $A \subseteq M$. Then $A^{\text{alg}} \cap M = \operatorname{acl}(A) = \operatorname{dcl}(A)$. (here there is at least one order)

There is a geometric structure with a good notion of dimension.

Denote by $M^{(i)}$ a fixed real closure of M with respect to $<_i$. Denote $a \mathop{\smile\!\!\!\!\!\!\,|}^{i}_{A} B$ if [usual thing] in the real closure.

Theorem 5.100 (Montenegro)**.** Forking equals dividing and $\mathop{\smile\!\!\!\!\!\!\,|} = \forall i < n \mathop{\smile\!\!\!\!\!\!\,|}^{i}$.

Theorem 5.101. Let M be a PRC field. Then [...]

[too fast]

[topology given by: intersection of intervals given by the different orders; the intervals may have endpoints in the real closures, but take the intersection with M; call these multi-semialgebraic sets as union of "multi-cells"]

[they are tame in some sense (maybe NTP_2)]

Isogeny: morphism of groups that is surjective and has finite kernel. [virtual isogeny]

[...massive omission]

Theorem 5.102. Let M be an ω-saturated model of T_{prc}. Let G be an M-defianble group with strong f-generics. Then there are G_1, K, H, H_1 such that

1. H is a multi-semialgebraic[2] group, M-defianble
2. $G_1 \leq G$, $H_1 \leq H$ of finite index and M-definable
3. $K \leq G_1$ is finite and central in G_1
4. G_1/K is defianbly ismorphic to H_1

[more massive omissions]

[2] Should be the same as "quantifier-free definable"

Chapter 6

Wednesday 5 July

6.1 Todor Tsankov – Model theory of measure-preserving actions

[slides] [joint with Thomás Ibarlucía]

Fix Γ a countable group, $(X, \mathcal{X}, \mu)$ a probability space, and $\Gamma \curvearrowright X$ by measure-preserving transformation.

Identify $A_1, A_2 \in \mathcal{A}$ if $\mu(A_1 \triangle A_2) = 0$. Thus $\mathcal{X}$ becomes a complete metric space with metric $d(A, B) = \mu(A \triangle B)$. $\mathcal{X}$ carries the structure of a Boolean algebra and Γ acts n $\mathcal{X}$ by isometric isomorphisms.

Example 6.1. The Bernoulli shift $\Gamma \curvearrowright 2^\Gamma$, defined as $(\gamma \cdot x)(\gamma') = x(\gamma^{-1}\gamma')$, where the measure is $(p\delta_0 + (1-p)\delta_1)^\Gamma$ for some $p \in (0, 1)$.

Example 6.2. Compact actions: if $\rho\colon \Gamma \to K$ is a homomorphism to a compact group and $L \leq K$ is a closed subgroup, ten $\Gamma \curvearrowright K/L$ by $\gamma \cdot kL = \rho(u)kL$. Te measure is the quotient of the Haar measure. E.g. for $\Gamma = \mathbb{Z}$ the irrational rotation on the circle, or the *odometer* (translation by 1 on $\mathbb{Z}_2$).

Definition 6.3. A substructure $\mathcal{Y} \subseteq \mathcal{X}$ is a closed, Γ-invariant subalgebra.

In ergodic theory, this is known as a *factor*, and is usually viewed dually, as a Γ-equivariant, measure-preserving map $X \to Y$.

Remark 6.4. Countable additivity comes automatically from continuity.

Recall that in continuous logic structures are complete metric spaces, $=$ is replaced by the metric, predicates are real-valued, bounded, uniformly continuous functions, connectives are continuous functions $\mathbb{R}^k \to \mathbb{R}$. We can take as a complete set of connectives the constants, addition and multiplication. Quantifiers are inf and sup, uniform limits of formulas are again formulas, and if $\varphi(x)$ is a formula with $|x| = n$ and M is a model, the interpretation of φ in M is a uniformly continuous, bounded function $M^n \to \mathbb{R}$, where the modulus of continuity and the bound can be determined syntactically from φ.

[my question: can one replace the metric with a uniform structure?]

The language for $(\mathcal{X}, \mu)$ is the following: use the language of Boolean algebras and add a predicate μ for the measure, which defines the metric. The axioms used are:

- Boolean algebra axioms
- μ is a probability
- μ is non-atomic (expressed as $\sup_A \inf_B |\mu(A \cap B) - \mu(A)/2| = 0$). Note that this does not say that B cuts A in half, just says that you can approximate this situation with arbitrary precision.

The resulting theory is ω-categorical and ω-stable.

To code the group action, add a function symbol for every $\gamma \in \Gamma$ and the axioms:

- each γ is an automorphism of $\mathcal{X}$.
- $\sup_A d((\gamma_1, \ldots, \gamma_n)A, A) = 0$ for every $\gamma_1, \ldots, \gamma_n$ such that $\gamma_1, \ldots, \gamma_n = 1_\Gamma$

Remark 6.5. If is enough to add symbols for a generating set for Γ. For instance, for $\mathbb{Z}$, one function symbol suffices.

Definition 6.6. $\Gamma \curvearrowright X$ is called *free* iff for every $\gamma \neq 1_\Gamma$ we have $\mu(\{x \in X \mid y \cdot x = x\}) = 0$.

So, in a sense, [every orbit looks like Γ?]

Axioms: for every $\gamma \neq 1_\Gamma$,

$$\sup_A \inf_B \mu(B \setminus A) + \mu(B \cap \gamma B) + |\mu(B) - \mu(A)/3| = 0$$

Calla $\mathrm{FR}(\Gamma)$ the resulting theory of free-measure preserving actions of Γ on a non-atomic probability space.

Definition 6.7. Γ is *amenable* iff there is a left-invariant, fiitely additive probability measure on Γ.

Example 6.8. Finite groups, abelian, or solvable groups are amenable. F_2 and any group containing it is not amenable.

Theorem 6.9 (Essentially Ben Yaacov-Berenstein-Henson-Usvyatsov)**.** Let Γ be amenable. Then $\mathrm{FR}(\Gamma)$ is a complete, stable theory that eliminates quantifiers. It is ω-categorical if and only if Γ is finite[1].

[1] Which is not interesting in the ergodic case.

(The considered the case of $\Gamma = \mathbb{Z}$, but the proof extends, using the machinery of Ornstein-Weiss, to amenable group actions)

(the theory of amenable groups is well-developed and it is probably difficult to get new results, so applying model theoretic techniques may be not fruitful)

Definition 6.10. $\Gamma \curvearrowright X$ is called *erogic* iff there are no non-trivial invariant sets: $\forall A \in \ mcX\ \forall \gamma \in \Gamma\ \gamma A = A \to \mu(A) \in \{0, 1\}$.

The ergodic decomposition theorem states that every measure, preserving action decomposes as an integral of ergodic components.

The Bernoulli action is always ergodic if Γ is infinite, and a a compact action $\Gamma \curvearrowright K/L$ is ergodic iff the morphism $\Gamma \to K$ used to define it has dense image.

Remark 6.11. Ergodicity is not, in general, an elementary property. THis can be seen, for instance, from the previous things (the problem is the formula $\gamma A = A$: you can only say it is *close* to A) (but the set of invariant A's should be type-definable).

Definition 6.12. A sequence $(A_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{X}$ is *asympotically invariant* iff

- $\mu(A_n)(1 - \mu(A_n)) \not\to 0$, and
- for every $\gamma \in \Gamma$, $\mu(\gamma A_n \triangle A_n) \to 0$

An action is *strongly ergodic* if it admits no asynptotically invariant sequences. Equivalently, if its ultrapower is ergodic.

Fact 6.13. Strong ergodicity is an elementary property, in the sense that it is invariant under elementary equivalence. This is because, by definition, it means that everything in the elementary class is ergodic.

Amenable groups do not admit strongly ergodic actions: Γ s amenable iff $\Gamma \curvearrowright 2^\Gamma$ is not strongly ergodic, iff no action [on what?] is strongly ergodic.

If Γ is non-amenable, no natural complete theories extending $\mathrm{FR}(\Gamma)$ are known.

Definition 6.14. An *existential formula* in one of the form $\inf_x(\varphi(x,),$ where φ is quantifier-free (x can be a tuple).

Definition 6.15. A system $\mathcal{Y}$ is *weakly contained* in $\mathcal{X}$ ($\mathcal{Y} <_w \mathcal{X}$) iff for every existential sentence φ, $\varphi^{\mathcal{X}} \leq \varphi^{\mathcal{Y}}$ (equivalently, if $\mathcal{Y}$ is a factor of an ultrapower of $\mathcal{X}$). Two systems are *weakly equivalent* iff they have the same existential theory.

The notion of weak containment was introduced by Kechris with a different, but equivalent, definition.

Fact 6.16. Some basic facts:

- If Γ is amenable, there is only one weak equivalence class of free Γ-actions
- An action is strongly ergodic iff the trivial action is not weakly contained in it
- (Abért, Weiss) The Bernoulli shift $\Gamma \curvearrowright 2^\Gamma$ is weakly contained in any free action of Γ
- (Bowen, Tucker-Drob) If Γ contains F_2, then there are uncountably many weak equivalence classes of free, strongly ergodic actions of Γ.

It is an open question whether this holds for every non-amenable Γ.

Theorem 6.17 (Ioana, Tucker-Drob)**.** If $\mathcal{X}$ is a comapct action, $\mathcal{Y}$ is strongly ergodic and $\mathcal{X} <_w \mathcal{Y}$, then $\mathcal{X}$ is a factor of $\mathcal{Y}$.

An analogous result had been previously shown for a finite $\mathcal{X}$ by Abért and Elek. (it is false if you remove strong ergodicity; for instance it is always false for amenable groups)

Corollary 6.18. If $\mathcal{X}$ and $\mathcal{Y}$ are compact, strongly ergodic, and weakly equivalent, then they are isomorphic.

This holds because compact actions are coalescent, i.e. every self-embedding is an automorphism.

Definition 6.19. Let $(X, \mathcal{X}, \mu)$ be a measure-preserving system and K a compact group. A *cocycle* is a measurable map $\alpha\colon \Gamma \times X \to K$ that satisfies the cocycle identity

$$\alpha(\gamma_1\gamma_1, x) = \alpha(\gamma_1, \gamma_2 \cdot x)\alpha(\gamma_2, x)$$

A *compact extension* of $\mathcal{X}$ is a system $(Y, \mathcal{Y}, v)$ where $Y = X \times K$, $\nu = \mu \times \mathrm{Haar}(K)$ and

$$\gamma \cdot (x, k) = (\gamma \cdot x, \alpha(\gamma, x)k)$$

Remark 6.20. A compact action is just a compact extension of the trivial action (on a one-point space).

Definition 6.21. A *distal action* is obtained by a transfinite series of compact extensions starting from the trivial action.

(this is equivalent to the usual definition of distality in topological dynamics because of Furstenberg's theorem) Distal actions first appeared in the work of Furstenberg on Szemerédi's theorem.

Beleznay and Foreman proved that for $\Gamma = \mathbb{Z}$, for any ordinal $\beta < \omega_1$, distal actions of distal rank[2] β exists.

[2] Lenght of the "canonical tower" used to produce it.

Example 6.22 (Parry, Walters)**.** Let $\mathbb{T}$ be the circle, $\alpha, \beta \in \mathbb{T}$ and $\theta \colon \mathbb{T} \to \mathbb{T}$. Define $S \colon \mathbb{T} \times \mathbb{T}^{n+1} \to \mathbb{T} \times \mathbb{T}^{n+1}$ by

$$S(z, w_1, w_2, w_3 \dots) = (z + \alpha, w_1 + \theta(z), w_2 + \theta(z + b), w_3 + \theta(z + 2\beta), \dots)$$

Then S is a distal transformation of rank 2 (compact extension of theirrational rotation $z \mapsto z + \alpha$ and α, β, θ, can be chosed so that it is ergodic.

Example 6.23. Let $\sigma \colon \mathbb{T} \times \mathbb{T}^n \to \mathbb{T} \times \mathbb{T}^n$ by

$$\sigma(z, w_1, w_2, \dots) = (z + \beta, w_2, w_3, \dots)$$

This commutes with S and defines a factor map which is not an iso. So there are ergodic, distal systems that are not coalescent.

Theorem 6.24 (Ioana, Tucker-Drob)**.** *If $\mathcal{X}$ is a distal action, $\mathcal{Y}$ is strongly ergodic, and $\mathcal{X} <_w \mathcal{Y}$, then $\mathcal{X}$ is a factor of $\mathcal{Y}$.*

However, the corollary from before does not immediately extend to distal actions because of the example of Parry and Walters.

Theorem 6.25. *Every strongly ergodic, distal system is coalescent. In particular, if two strongly ergodic, distal systems are weakly equivalent, then they are isomorphic.*

Recall that B is definable over A if the distance predicate $d(\cdot, B)$ is definable over A. Say that a is algebraic over A iff it belongs to a compact set definable over A. THe acl is the obvious thing, and the *existential algebraic closure* is [...]

Theorem 6.26. *Let $\mathcal{X}$ be a strongly ergodic system and Γ be a distal factor for $\mathcal{X}$. Then $\mathcal{Y} \subseteq \operatorname{acl}^{\exists}(\emptyset)$.*

There are two different proofs: the first by induction on the distal rank of $\mathcal{Y}$. The second one uses a different characterisation of distality and a bit of stability theory.

Corollary 6.27. *Every strongly ergodic, distal system is coalescent.*

Proof. Let $\mathcal{X}$ be strongly ergodic and distal and let $\sigma \colon \mathcal{X} \to \mathcal{X}$ be an emebedding. Then σ maps 0-sets of existential formulas to themselves and an isometric injection of a compact set into itself is automatically surjective. □

Question 6.28. Do strongly ergodic, distal, non-compact action exist?

Definition 6.29. A group Γ has Kazhdan's property (T) iff all of its ergodic actions are strongly ergodic. Or, equivalently, if ergodicity of measure-preserving actions of Γ is an elementary property.

Example 6.30. SL$(3,\mathbb{Z})$ or, more generally, lattices in high rank, real simple Lie groups. Also, random groups in the sense of Gromov.

Theorem 6.31. Let Γ have property (T). THen every distal Γ-action is compact.

This theorem was also proved by Chifan and Peterson using methods from operator algebras.

6.2 Rizos Sklinos – Some model theory of the free group

[slides; a lot missing; look at the slides]

[definition of free group] Notation: if $\mathbb{F}$ is free on S, then S is called a basis and its cardinality is called rank.

$\mathbb{F}$ is free if it admits a free action [definition] without inversion on a tree (nonoriented connected graph with no cycles)

It follows that

Theorem 6.32. Subgroups of free groups are free.

Question 6.33 (Tarski)**.** Do nonabelian free groups share the same common first-order theories?

Note that it is false for abelian groups

Question 6.34. If $[F_n, F_n]$ definable in F_n (note that the quotient is $\mathbb{Z}^n$)

Theorem 6.35 (Sela 2001, Kharlampovich-Miasnikov)**.** Non-abelian free groups share to thesame common first-order theory.

As a mapper of fact, the chain $\mathbb{F}_2 \leq \mathbb{F}_3 \leq \ldots$ is elementary.

Sela described all finitely generated models of the first-order theory of the free group. He called them *hyperbolic towers.*

Theorem 6.36. The theory of the free group is nonequational.

Theorem 6.37. The theory of the free group is stable.

[my question: how does forking look like there?]

Theorem 6.38. [missed it]

Theorem 6.39 (Poizat)**.** $\mathbb{F}_\omega$ is connected and not superstable.

Theorem 6.40 (Pillay)**.** An element of a nonabelian free group is generic if and only if it is primitive, i.e. it is part of some basis. Any maximal independent set of realisations of the generic type in $\mathbb{F}_n$ is a basis of $\mathbb{F}_n$.

Theorem 6.41 (Pillay, Sklinos)**.** The generic type has infinite weight.

Theorem 6.42 (Louder-Perin-Sklinos)**.** There is a finitely generated $G \vDash T_{\mathrm{fg}}$ and two (finite) maximal indepdent sequences of relaisations of the generic type in G of different length.

Theorem 6.43. [. . .]

Theorem 6.44 (Perin-Sklinos, Ould Houcine)**.** $\mathbb{F}_n$ is homogenous.

As a matter of fact every nonabelian free group is strongly $\aleph_0$-homogeneous

Theorem 6.45 (Sklinos)**.** [something like: no [something] is $\aleph_1$-homogenous.

Remark 6.46. Most π_1 of surfaces are not homogeneous

Theorem 6.47 (Perin-Sklinos)**.** Let $\mathbb{F}$ be a nonabelian free group and $b, c, \subseteq \mathbb{F}$. Then $b \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} c$ iff $\mathbb{F}$ admits a free splitting as $B * C$ with $b \subseteq B$ and $c \subseteq C$.

Theorem 6.48 (Perin-Sklinos)**.** Let $\mathbb{F}$ be a nonabelian free group and $b, c, A \subseteq \mathbb{F}$. Ten $b \mathop{\smile\!\!\!\!\!\!\;|}_{A} c$ if and only if [picture]

Theorem 6.49 (Pillay)**.** The free group is not CM-trivial, i.e. it is 2-ample.

Theorem 6.50 (Ould Houcine-Tent/Sklinos)**.** The free group is n-ample for all $n < \omega$.

Remark 6.51. The main tool for confirming hte algebraic conditions of ampleness is Thurston's pseudo-Anosov homeomorphisms.

Theorem 6.52 (Byron-Sklinos/Sklinos)**.** No infinite field is interpretable in the free group.

Theorem 6.53. Let X be a definable set ina nonabvelian free group $\mathbb{F}$. Then either X is internalt o a fintie set of centralisers of non trfivial elements, or it cannot be given definably the structure of an abelian group

Theorem 6.54. Centralisers of elements in onnabelian fgree groups a re pure groups, i.e. the induced[. . .]

[. . .]

Theorem 6.55 (Sklinos)**.** The free group has nfcp.

Theorem 6.56 (Sela)**.** The free group is nonequational.

Theorem 6.57 (Müller-Sklinos)**.** No free product, except $\mathbb{Z}_2 * \mathbb{Z}_2$ is equational.

Theorem 6.58 (Sela)**.** Any free product of stable groups is stable.

This gives many examples of natural stable structure which are not equational (people used to think the natural examples would be equational).

Question 6.59 (Malcev)**.** Is $[\mathbb{F}_n, \mathbb{F}_n]$ definable in $\mathbb{F}_n$?

Theorem 6.60 (lot of people)**.** Any proper definable subgroup of a nonabelian free group is cuclic.

Conjecture 6.61. The only interpretable [groups?] are the "obvious" ones.

Fact 6.62. let G be a free product of nonanbelian groups and surface groups $\pi_1(\Sigma)\ with(\chi(\Sigma) < -1)$. Then any finite index subggroup of G is elementarily equivalent to G.

[...]

Theorem 6.63. Let $\varphi(x)$ be a formula over $\mathbb{F}_n$. SUppose $\mathbb{F}_n \neq \varphi(\mathbb{F}_\omega)$. Then $\varphi(x)$ is not superstable.

Conjecture 6.64. The converse holds.

Theorem 6.65. The free gorup admits quantifier elimination up to boolena combination sof $\forall\exists$ formulas

Theorem 6.66. The free group is not model complete.

Question 6.67. Does the theory of the free group admit a model companion?

6.3 Silvain Rideau – Imaginaries in pseudo-p-adically closed fields

[slides][joint with Samaria Montenegro]

Definition 6.68. A valuation is p-adic if the residue field is $\mathbb{F}_p$ and p has minimal positive valuation. A field extension $K \leq L$ is *totallu p-adic* if every p-adic valuation of K can be extension can be etended to a p-adic vlauation of L. A field K is pseudo-p-adicallu closed iff it is existentially closes as a ringin every regular totally P-adic extension. A field is bounded iff it has finitely many extensiolns of any given degree.

Proposition 6.69 (Montenegro)**.** let K be a bounded preudo-p-adically closed field. There are finitely many p-adic valuations on K and they are definable in the ring language.

Let (K, v) be a valued field.

Definition 6.70. The *geometric sorts* are $S_n = \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$ and $T_n = \mathrm{GL}_n(K)/\mathrm{GL}_{n,n}(\mathcal{O})$. The geometric language has sorts $F, S_n and T_N$ for all $n \geq 1$. It also contains the ring language on F, the canonicalprojections $s_n \colon \mathrm{GL}_n(F) \to S_n$ and $t_n \colon \mathrm{GL}_n(F) \to T_n$

Remark 6.71. $S_1 \cong \Gamma$ and s_1 is the valuation.

Theorem 6.72 (HHM)**.** Algebraically closed valued fields eliminate imaginaries in the geometric language.

Theorem 6.73 (HMR)**.** $\mathbb{Q}_p$ eliminates imaginries in the geometric language.

Let K be a bounded pseudo-p-adically closed field with n p-adic valuations $(v_i)_{i\leq n}$. Let L_i be n copies of the geometric language, with sorts G_i, sharing the sort F. Let $K_0 \prec K$, $L = \bigcup L_i \cup K_0$ and $T = \mathrm{Th}_L(K)$. Let $M \vDash T$, $\bar{M}_i$ be the algebraic closure of M with an extension of v_i and M_i the p-adic closure of M inside $\bar{M}_i$. (it is the henselisation because the value group is [?])

Remark 6.74. Let $U_i \neq \emptyset$ be v_i-open, ten $\bigcap U_i \neq \emptyset$. In a sense, the valuations are independent.

Proposition 6.75. Let $K_0 \subseteq A \subseteq F(M)$ and $s_i, t_i \in S_{i,n}(M)$. If $\forall i s_i \equiv^{M_i}_{L_i(A)} t_i$ then $(s_i)_{i\leq n} \equiv^{M}_{L(A)} (t_i)_{i\leq n}$.

This is a nice orthogonality result, but it only talks of the generic sorts, not of the field, which is the most important part.

Let $A \subseteq M^{\mathrm{eq}}$ containing $\bigcup_i G_i(\mathrm{acl}^{\mathrm{eq}}_M(A))$.

Proposition 6.76. Let $c \in F(M)$. Then for all i there is a $G_i(A)$-invariant $L_i(\bar{M}_i)$-type p_i such that $\mathrm{tp}^M_L(c/A) \cup \bigcup_i p_i$ is consistent.

Proposition 6.77. Let $c \in F(M)$ and $d \in G_i(\mathrm{acl}^{\mathrm{eq}}_M(Ac))$ be some tuples. Assume $\mathrm{tp}^{\bar{M}_i}_{L_i}(c/\bar{M}_i)$ is $G_i(A)$-invariant, then so is $\mathrm{tp}^{\bar{M}_i}_{L_i}(d/\bar{M}_i)$.

Corollary 6.78. Let $c \in F(M)$ be some tuple. Then for all i there is $G_i(A)$-invariant $L_i(\bar{M}_i)$-type p_i such that $\mathrm{tp}^M_L(c/A) \cup \bigcup_i p_i$ is consistent.

(in the three things above the c's are in an elementary extension) (this uses elimination of imaginaries for ACVF, but can be used to give an alternate proof for $\mathbb{Q}_p$.)

Let $(L_i)_{i\in I}$ be languages, with sorts R_i, sharing a comon dominant sort D, and let $L \supseteq \bigcup_i L_i$. Let T_i be L_i-theories and $T \supseteq \bigcup_i T_{i,\forall}$ be an L-theory. Let $M \vDash T$ and $M \subseteq M_i \vDash T_i$ be sufficiently saturated and homogeneous. For all $C \leq M^{\mathrm{eq}}$ and all tuples $a, b, \in D(M)$, write $a \mathop{\smile\!\!\!\!\!\!|}_C b$ iff there are $R_i(A)$-invariant $L_i(M_i)$-types p_i with $a \vDash \bigcup_i p_i \mid R_i(C) b$

Proposition 6.79. Assume:

1. for all $A = \operatorname{acl}^{\mathrm{eq}}_M(A) \subseteq M^{\mathrm{eq}}$ and a tuple $c \in D(M)$, there is $d \equiv ?M_{L(A)}c$ with $d \mathop{\smile\!\!\!\!\!\mid}_A M$
2. For all $A = \operatorname{acl}_M(A) \subseteq M$ and $a, b, c \in D(M)$ tuples, if $b \mathop{\smile\!\!\!\!\!\mid}_A a$, $c \mathop{\smile\!\!\!\!\!\mid}_A ab$, $A \equiv^M_{L(A)} b$ and $ac \equiv^{M_i}_{L_i(R_i(A))} bc$, for all i, then there is d such that $db \equiv^M_{L(A)} da \equiv^M_{L(A)} ca$.

Then T weakly eliminates imaginaries.

In pseudo-p-adically closed fields, Condition 2 follows from a more general result [highly technical; see slides]

Theorem 6.80. The theory T eliminates imaginaries.

Remark 6.81. Remarks:

- Coding finite sets is not completely obvious (you have to deals with stuff living in the geometric sorts of different valuations)
- Since the valuations v_i are discrete, $T_{i,n}$ is coded in $S_{i,n+1}$.
- Let $\mathcal{O} = \bigcap_i \mathcal{O}_i$. We ahve a bijection

$$\prod_i S_{i,n} = \prod_i \operatorname{GL}_n(F)/\operatorname{GL}_n(\mathcal{O}_i) \simeq \operatorname{GL}_n(F)/\operatorname{GL}_n(\mathcal{O})$$

Chapter 7

Thursday 6 July

7.1 Franz-Viktor Kuhlmann – Pushing back the barrier of imperfection

[slides]

[perfect fields][defect][only happens in positive characteristic]

Defect/imperfection gets in the way of resolution of singularities, and of model theory of valued fields.

Example 7.1 (Abhyankar, '50)**.** Something that does not lie in the Puiseux eries fierd ofver $\mathbb{F}_p$ even if it is algebraic over [$\mathbb{F}((t))$?].

This also shows that a certain AKE-type result does not hold for $\mathbb{F}_p((t))^{\frac{1}{p^\infty}}$.

The common underlying problem for local uniformisation ad the model theory of valued fields is the structure theory of valued function fields.

For model theory, the proof of AKE-type results is reduced to embedding lemmas for finitely generated extensions of valued fields; these are valued function fields. After having dealt with embeddings of rational function fields, usually the only tool available for extending these to the full function fields is Hensel's Lemma. Also, sometimes you can you uniqueness of maximal immediate extensions, but this only works in Kaplanski fields.

Local uniformisation: eliminating one singularity at a time by passing to a new, birationally equivalent variety (i.e. they have the same function fields). Zariski introduced the use of valuations on the function field of te variety in order to trace the point on thenew variety which corresponds to the original singular point. We want the new point to be smooth, i.e. that the implicit function theorem is satisfied. This is essentially the same as saying the point satisfies Hensel's lemma.

Ramification is the valuation theoretical expression of the failure of it. So we wish to eliminate ramification in a given valued function field $(F \mid K, v)$. This means to find a transcendence basis T such that F lies in the *absolute inertia field* (strict henselisation) of $(K(T), v)$. In other words, $(F^h \mid K(T)^h, v)$ of respective henselisations shall satisfy:

1. it is defectless
2. the value group extension is trivial
3. the residue field extension is separable.

In positive characteristic, the defect gets in the way of solving this.

It is not know whether resolution of singularities, or even local uniformisation, or even elimination of ramification, holds in positive characteristic for dimensions greater then 3.

Algebraic geometers do not believe resolution of singularities to be true, but may local uniformisation is.

Definition 7.2. A henselian valued field (K, v) is *tame* iff every finite extension $L \mid K$ satisfies the following conditions:

(Ti) The ramification index $(vL : vK)$ is not divisible by $\operatorname{char} Kv$

(Tii) The residue field extension $Lv \mid Kv$ is separable.

(Tiii) The extension $(L \mid K, v)$ is defectless.

Remark 7.3. All tame fields are perfect, but not necessarily Kaplansky.

Theorem 7.4 (Kuhlmann)**.** Tame fields satisfy model completness and decidability relative to the elementary theories of their value groups and residue fields. If $\operatorname{char} K = \operatorname{char} Kv$, then also relative completeness holds.

Example 7.5. $\mathbb{F}_p((t))^{\frac{1}{p^\infty}}$ is not tame, as it does not satisfy the last condition.

Crucial for the proof of the theorem above is this, which eliminates ramification in particualr function fields over tame fields

Theorem 7.6 (Kuhlmann)**.** Let (K, v) be a tame field, $(F \mid K, v)$ a valued function field with $vF = vK$ and $Fv = Kv$. [...]

THe Henselian Rationality Theorem is also a crucial ingredient inthe proof that every valued function field admits local uniformisation if one takes a finite extension of the function field (alteration) into the bargain (joint with H. Knaf). This is a local version of de Jong's *resolution by alteration*. But here it can be showwn that, in addition, the extension can be taken to be Galois, and the proof is purely valuation theoretical.

Question 7.7. Can we do without alteration? Can we get from the tame fields to $\mathbb{F}_p((t))$ without [...]?

We now see three examples where we hit the barrier of imperfection.

Definition 7.8. A Henselian valued field is *separably tame* iff every finite *separable* extension satisfies te conditions of definition 7.2.

These are somehow nice [theorem by K. Pal and Kuhlmann], but they are not necessarily perfect. They are dense in their perfect hulls.

Question 7.9. Assume that $(F \mid K, v)$ is a valued functionfield wit v trivial on K and $Fv = K$, i.e. F admits a K-rational place. Does it follow that $K \prec_\exists F$?

FOr this to hold for all such F, the field K needs to be rich enough. Calla afield *large* if for every smooth curve over K the set of rational points is infinite as soon as it is nonempty, or equivalently if $K \prec_\exists K((t))$. For instance, if a field satisfies and implicit function theorem (which is te case if it is henselian wrt some non trivial valuation), ten it is large.

Theorem 7.10 (Kuhlmann)**.** If K is perfect, then the following are equivalent.

1. K is large
2. [something]
3. [something else]

What if K is large, but not perfect?

Theorem 7.11. Let K be a large field and $F \mid K$ an algebraic function field with a K-rational place P. If $(F \mid K, P)$ admits local uniformisation, then $K \prec_E F$.

[I guess the last is one of the conditions equivalent to being perfect, in the previous theorem]

Definition 7.12 (Yu. Ershov, 2003)**.** A valued field is *extremal* iff for all n and polynomials $f \in K[X_1, \ldots, X_n]$, the set $\{vf(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in K\}$ has a maximal element (which may be ∞).

Ershov stated that all Laurent series fields $k((t))$ with their t-adic valuations are extremal. Bu Starchenko showed that $\mathbb{R}((t))$ is not. [should imply being algebraically closed] Kuhlmann corrected the definition:

Definition 7.13 (Yu. Ershov, 2003)**.** A valued field is *extremal* iff for all n and polynomials $f \in K[X_1, \ldots, X_n]$, the set $\{vf(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathcal{O}\}$ has a maximal element (which may be ∞).

With the new definition, Kuhlmann proved Ershov's claim. This means that extremality is an elementary property of $\mathbb{F}_p((t))$.

Question 7.14 (Open)**.** Is the following axioms system complete? (K, v) is an extremal valued field of characteristic p with value group a $\mathbb{Z}$-group and residue field $\mathbb{F}_p$

Note that "extremal" implies henselian and [something else]

Theorem 7.15 (Anscombe, Azgin, Pop, Kuhlmann)**.** Let (K, v) be a nontrivially valued field. If it is extremal, then it is henselian and defectless, and

1. vK is a mbZ-group, or
2. vK is divisible and Kv is large.

Conversely, if (K, v) is henselian and defectless, and

1. $vG \simeq F$ or vK is a $\mathbb{Z}$-group and $\operatorname{char} Kv = 0$, or
2. v_K is divisible and Kv is large and perfect,

then (K, v) is extremal.

Completing the characterisation above would mean pushing the barrier of imperfection, extremal fields are the building blocks of all $\aleph_1$ saturated valued fields:

Theorem 7.16. Let (K, v) be any $\aleph_1$ valued field. Asume that the place P associated with v is decomposed as $P = P_1P_2P_3$ with P_2 of rank 1. Then (KP_1, P_2) is extremal and large, and its value groups is isomorphic either to[...]

M. Temkin proved that local uniformisation can also be achieved after a finite purely inseparable extension of the function field. This any way just means that the defect has been "stowed away" in a separable or a purely inseparable extension. However, this tells us something about the type of defect that has not been mastered.

Definition 7.17. Let (K, v) have characteristic p. Call a Galois defect extension of degree p *dependent* iff [...]

[big omission]

What is a suitable analogue of the classification of defect extensions in mixed characteristic?

A modern tool for transferring information between the mixed and the positive characteristic case are the *perfectoid fields*, via the *tilting construction*. Can information about defects also be transferred?

Turns out that the perfectoid fields are not suitable for the purpose (they need to be complete, so second order). [...]Instead, one uses

Definition 7.18. A valued field (K, v) is *deeply ramified* if either $\operatorname{char} Kv = 0$, or $\operatorname{char} Kv = p > 0$, the value $v(p)$ is not the smallest positive element in vK and the Frobenius homomorphism on $\hat{\mathcal{O}}/p\hat{\mathcal{O}}$ is surjective, where $\mathcal{O}$ is the completion [. . .]

[. . .]

Theorem 7.19 (Kuhlmann)**.** Over deeply ramified fields all Galois defect etensions of prime degree are independent.

Theorem 7.20 (Kuhlmann)**.** All tame fields are deeply ramified (but not conversely).

Conjecture 7.21. The Henselian Rationality Theorem also holds over henselian deeply ramified ground fields in place of tame ground fields, provided they are relatively algebraically closed in the function field.

Can interesting model theoretic results about henselian deeply ramified fields be proven in suitable extensions of the language of valued fields?

So

1. Separably tame fields: defect appears only witing pureley inseprable extensions, and they are contained in the completion. This is done.
2. Extremal fields: defectless, in general not perfect. Include $(\mathbb{F}_p((t)), v_t)$. To be done
3. Henselian deeply ramified fields: [. . .] to be done.

Valuation theory home page: `http://math.usask.ca/fvk/Valth.html`. Preprint server, conference announcements, ecc.

7.2 Ehud Hrushovski – More on the product formula

[slides][joint with I. Ben-Yaacov]

The fundamental theorem of arithmetic relates the real norm $|x|$ with all p-adic norms by $\int_p v_p(x)\,\mathrm{d}m(v) = 0$, where m gives each v_p weight $\log_p$ and v_∞ weight 1 (here $v_\infty(x) = -\log|x|$).

In this form, the formula is valid for any $n \in \mathbb{Q}^*$, in fact for any number field and global function field. In $L_{\omega_1,\omega}$, it axiomatises these precisely (Artin-Whaples 1940). We aim for a first-order, continuous-logic axiomatisation.

The language has a field sort $(F, +, \cdot, 0, 1)$ and a sort $(\mathbb{R}, +, <)$. Continuous logic is used to insist that the latter always has the standard interpretation. On F, terms a polynomials over $\mathbb{Z}$; equality is a $\{0, 1\}$ valued relation as usual.

On $\mathbb{R}$, the "tropical terms" are terms in the signature $\{+, \min 0, \alpha \cdot x \mid \alpha \in \mathbb{R}\}$. Or allow theuniform closure of this, i.e. all continuous, positively homogeneous functions $\mathbb{R}^m \to \mathbb{R}$.

Basic symbols I_t: a symbol I_t for each tropical term t; to be interpreted as a function $(F^*)^n \to \mathbb{R}$. Local interpretation of I_t: let v be a valuation with value group $\mathbb{R}$, *or* a place $p \to \mathbb{C}$ with $v(x) = -\alpha \log|x|$. Interpret $I_t^v(x)$ as $t(vx_1, \dots, vx_n)$.

Remark 7.22. The discreteness of F is unusual in continuous logic. It reflects a deep fact: discretenss of $\mathbb{Q}$ in the adeles, = discretenss of $\mathbb{Z}$ in $\mathbb{R}$. An algebraic number can get close to 0 in the real topology, or in any p-adic topology, but not in all of these topologies at once.

Axioms GVF for globally valued fields:

1. $(F, +, \cdot)$ is a field
2. The I_t are compatible with permutations of variables and dummy variables
3. Linearity of I
4. [something else, maybe some completeness]
5. Product formula: $I_x = 0$ for $x \neq 0$

Proposition 7.23. Let $F \vDash \mathsf{GVF}$. Then there is a measure m on a space Ω_F of absolute values of F such that [...]

Plan: let X be a smooth projective variety over a globally valued field F. A formula on X is a combination of:

1. Adelic formulas
2. Néron-Weil character
3. A positive affinemap on a certain torsor of $NS(X')$, $X' \to X$ birational

We will see the first two points. This, along with finite dimensionality of $NS(X)$, will suffice for qf stability.)

7.2.1 Adelic formulas

What formulas would you use to describe probaiblity measures μ on $\mathbb{C}$, $\mathbb{C}^n$, or $X(\mathbb{C})$? $\int \varphi \, d\mu$, for varuous test functions φ on X. The GVF language includes precisely this, with respect to the measure on $X(\mathbb{C})$. What measure? An archimedean valuation v on $\mathbb{Q}(x_1, \dots, x_n)$ is the same as a point $(\alpha_1, \dots, \alpha_n)$ of $\mathbb{C}^n$ (namely, $v(x_i) = -\log|\alpha_i|$). SO a GVF structure on the function field $\mathbb{Q}(X)$ includes the data: a conjucation invariant measure on $X(\mathbb{C})$. What test functions? $t(1, x_1, \dots, x_n)$, with $t(u, \dots, x_n)$ a tropical term., Take t such that if $u = 0$ then $t = 0$. u is used to de-homogenise t.

Let T'_r be the set of temrs $t(u_1, \dots, x_m)$ such that if $v(u) = 0$ then $t = 0$. Let $L[a]$ consist of formulas

$$\frac{1}{ht(a)} \int t(va^+, vx_1, \dots, vx_m)$$

with $t \in Tr'$. $\bigcup\{L(a) \mid a \in K\}$ is the K-adelic part of the language, over K.

Semantics: a GVF structure on K *includes* a probability measure on the space of valuations for K with $v(a) = 1$ (these are a set of representatives for the valuations v with $v(a) > 0$)

$L[a]$) language of expectation operators for such measures.

Example 7.24. [...]

[missing stuff]

Example: height on $\mathbb{P}^n$, defined as $\mathrm{ht}(x_0 : \dots : x_n) := \int -\min_{i=0}^n v(x_i)\,\mathrm{d}v$. This is well-defined in projective coordinates.

Example 7.25. For $a = (m_0 : \dots : m_n \in \mathbb{P}^n(\mathbb{Q})$, $\mathrm{ht}(a) = \max \log|m_i|$, when the m_i are relatively prime integers.

If $g \colon V \to \mathbb{P}^n$ is a projective embedding, $\mathrm{ht}_g(x) := \mathrm{ht}(g(x))$. For $x \in \mathbb{A}^1$, $\mathrm{ht}(x) = \mathrm{ht}(x : 1)$.

Example 7.26. For $x \in \mathbb{Q}^{\mathrm{alg}}$, $\mathrm{ht}(x) = 0$ iff x is an algebraic integer and every Galois conjugate lies on the unity circle. This is iff x is a root of unity (Kronecker).

Let $\mu = \mu_{G_m}$ be the $(\bigwedge)$-definable subset of G_m defined by $\mathrm{ht}(x) = 0$.

Theorem 7.27. The induces qf structure on μ is that of a pure group.

(in the purely non-archimedean case, the induced qf structure on μ is that of a pure field)

Corollary 7.28 (Bilu)**.** A sequence of Galois orbits fo algebraic integers, of heights approaching 0, is equidistributed on $\mathbb{C}$ along the circle $|z| = 1$.

Proof. Take the ultraproduct of $(\mathbb{Q}^a, a_i)$, with a_i in the i'th orbit, to obtain (K, a) with $\mu(a) = 0$, a non-algebraic. On the other hand, take the ultraproduct of $(\mathbb{Q}^a, \omega_i)$, with ω_i a primitive i'the root of 1, to obtain (K, a'). Then a, a' are non-algebraic elements of μ; so $a \equiv a'$. This includes in particular the complex measure. ☐

Let $K \models \mathsf{GVF}$, A an abelian variety over K. Let $[m] \colon A \to A$ denote multiplication by m. Fix an embedding $DD \colon A \to \mathbb{P}^N$ such that $[-1]$ is linear.

Theorem 7.29 (Weil-Néron-Tate). The GVF formulas $4^{-n}\operatorname{ht}_D \circ [2^n]$ converge uniformly to a limit denoted $\hat{h}_D$. We have:

- $\operatorname{ht}_D - \hat{h}_D$ is bounded, and
- $\hat{h}_D$ is a positive semi-definite quadratic form.

Bounded means: in any GVF extending K.

Proposition 7.30.

1. There is a unique maximal ∞-defianble subgroup μ of A (of bounded height)
2. For any D as above, $\mu = \{x \mid \hat{h}_D(x) = 0\}$
3. A/μ carries a natural (hyper) definable $\mathbb{R}$-Hilbert space structure [more stuff]

[...]

$0 \to \mu_A \to A \to \hat{A} \to 0$. $\hat{A}$ is far from being a pure Hilbert space. Nevertheless we will see that the HIlbert space structure plays a critical role.

μ_A is conjecturally a pure module over $\operatorname{End}(A)$, in GVFs containing $\mathbb{Q}[1]$. Close to theorems of Szpiro, Zhan, Ullmo, Gubler,... around equidistribution and the Bogomolov conjecture.

7.2.2 The Néron-Weil character: curves

Let X be a smooth, irreducible projective curve over a GVF F, 0 on $X(F)$. Let p be a qf GVF type on X, over F. We will define a homomorphism $NW_p\colon J(F) \to \mathbb{R}$ on the Jacobian J. There is a projective embedding g_0 and a hyperplane intersection $g_0(X)$ in $6[0]$. There is a projective embedding g_b and a hyperplane intersection $g_0(X)$ in $6[0] + [b]$. $\varphi_b = \operatorname{ht}_{g_b} - \operatorname{ht}_{g_0}$ is linear up to bounded; $\varphi(2x) - 2\varphi(x)$ is bounded.

The GVF formulas $2^{-n}\operatorname{ht}_D \circ [2^n]$ covnerge unifmroluy to a limit denoted $\hat{h}_b$. Define $NW_p(b) = \hat{h}_b(p)$.

[missing a lot of stuff]

Theorem 7.31. Any quantifier-free GVF type on a curve X over F is determined by:

1. The height (of the first nontrivial coordinate)
2. The F-adelic qf type (varlues of F-adelic formulas), and
3. The Néron character NW_p.

An extension holds for any smooth projective variety X. It allows defining a cnonical extension of a qf type over F to one over a GVF $K \geq F$.

1. Formulas over F
2. Canonical extension of local types
3. in Hilbert spaces
4. Mass zero to any exceptional divisor strictly over K

[proof, using the product formula]

Corollary 7.32. Let $F \leq K$. Formulas on X over K are uniform limits of [...]

Theorem 7.33. The theory GVF is qf stable.

This means that if (a_i, b_i) is a qf-indiscernible sequence and φ is a formual, then $\pi(a_1, b_2) = \varphi(b_1, a_2)$.

Ingredients: for $\mathbb{R}$ and $\mathbb{C}$, as treated in cotninuous logic, an indiscernible sequence is constant. In valued fields an indiscernible sequence (or invariant type) orthogonal to the value group is an indiscernible set (stably dominated). Here we decree that no type increase the value group, which is $\mathbb{R}$.

Randomisations of VF relative to F: [Ben-Yaacov/Keisler randomisation], but relative to ACF, i.e. for any formula of the language of rings impose $[\varphi] \in \{0, 1\}$.

[...]

Hilbert spaces: first example truly belonging to continuous logic. Independent= pairwise(!) orthogonal (over base) (like pure sets!). Indiscernible subspaces= orthogonal over their intersection (1-based).

[proof of qf stability] [more stuff]

7.3 Predrag Tanović – Colored orders with equivalence relations

[joint with Slavko, Dejan]

There is an old paper by Rubin, 1974, called *Theories of linear order*. The talk is about this paper and related results.

Theorem 7.34 (Rubin)**.** Let T be a complete countable theory of coloured orders. Then

1. $I(T, \aleph_0) \in \omega \setminus \{2\} \cup \{2^{\aleph_0}\}$
2. If the language is finite, $I(T, \aleph_0) \in \{1, 2^{\aleph_0}\}$

In order to do this, Rubin had to understand the countable sets, somehow, but he did not give an explicit description of definable sets. Poizat, in his book, gave an explicit description of 2-types: if $a < b$, then $\text{tp}(a, b)$ is described by $\text{tp}(a)$, $\text{tp}(b)$, $\{\text{tp}(c_1)\,\text{tp}(c_2), \ldots, \text{tp}_{c_n} \mid a < c_1 < \ldots < c_n < b\}$.

In *On dp-minimal orders*, Simon gave the following description.

Fact 7.35. Start with a linear order $(M,<)$ and add all definable unary predicates P_i and all monotone[1] binary relations R_j. Then this structure eliminates quantifiers.

Why colored orders, as opposed to just pure orders? The class of all of them is in some sense as complicated as the class of colored orders. For instance every colored order is a homomorphic image of a pure order, in the following sense. If $(M,<,P)$ is an order with a unary predicate, one can blow up every point in P to something like $1+\mathbb{Z}+1$ and every point not in P to something like $2+\mathbb{Z}+2$. There is a convex equivalence relation on the resulting linear order; taking the quotient is an homomorphism of linear orders, and the coloured structure can be recovered.

The above was done in the 1960's or before. We make another trivial observation:

Remark 7.36. Coloured orders (CO) are also, in some sense, of the same complexity as coloured orders with convex equivalences (COCE).

Theorem 7.37. Let $(M,<,P_i,E_j,s_k)$ where the P_i name all the $(M,<)$-defianble unary predicates, the E_j name all $(M,<)$-definable equivalence relations, and s_j is a successor relation on the classes of E_j. Then this structure has quantifier elimination.

Corollary 7.38. Any definable set in a CO is a Boolean combination of $\emptyset$-definable unary predicates, classes of $\emptyset$-definable convex equivalences, and intervals.

Example 7.39. In ω^2 we have a definable function picking the next limit point. This is the minimal point in the next class of a $\emptyset$-definable equivalence relation.

What could be a description of 2-types? Suppose $a \vDash p$ and $b \vDash q$, and the loci of P and q are somehow twisted (defined as "interval types" in Rubin), so they are not weakly orthogonal. We have some convex defianble equivalences, and there is a [definable?] order on them (inside the type): on $\langle E_i \mid i \in I\rangle$, if $i \neq j$ then either each E_i- class is spit into classes of E_j or the other way around [?]

There is something called *testing formula*, some kind of stable embeddedness of convex sets with a long definition.

Fact 7.40 (Linear Binarity (LB), (someone))**.** Whenever $a_1 < \ldots < a_n$,then $\bigcup_{i<n} \operatorname{tp}(a_i,a_{i+1}) \vdash \operatorname{tp}(\bar{a})$. (equivalently, every formula is equivalent to a Boolean combination of formulas with at most two free variables).

$(\mathbb{Q},<,E_2)$ has LB, but the one with three classes $(\mathbb{Q},<,E_3)$ has not.

[1] I.e. the fibers are non-decreasing: $R \subseteq A \times A$ is monotone if $x' \leq xRy \leq y' \to x'Ry'$.

Theorem 7.41. Suppose $(\mathfrak{U}, <, \ldots)$ is a saturated linearly ordered structure satisfying the *SLB condition (strong linear binarity)*, i.e. for every convex $C \subseteq \mathfrak{U}$ and any $f \in \mathrm{Aut}(\mathfrak{U})$ such that $f(C) = C$, the following happens. Any g defined as

$$g(x) = \begin{cases} f(x) & x \in C \\ x & x \notin C \end{cases}$$

is in $\mathrm{Aut}(\mathfrak{U})$. Then $(\mathfrak{U}, <, P_i, E_i)$ (expansion with all definable unary predicates and convex equivalences) is definitionally equivalent to the original structure.

So in a sense, if a structure has SLB, then it is in some sense derived from a pure linear order.

This characterises SLB structure. Can one characterise the structures that have LB? (actually LB is not that useful)

Definition 7.42. *Condition (F)* is the following statement. For any $\varphi(x, y)$ and $a \in \mathfrak{U}$ there are only finitely many φ-types with parameters from $(-\infty, a]$ that are realised in $[a, +\infty)$. (in the sense that every element of the tuple comes from that interval).

Fact 7.43. (SLB) $\Rightarrow$ (LB) $\Rightarrow$ (F)

There is a (not as nice as the one for SLB) characterisation of (F) in terms of *almost convex* equivalences (finite index in the convexisation) and something else.

We now switch to Vaught's conjecture for weakly o-minimal T. The results are:

1. Nearly reduced to the binary case.
2. Proved the binary case.

Slavko spoke about the proof of the binary case.

Theorem 7.44. Let T be weakly o-minimal with $I(T, \aleph_0) < 2^{\aleph_0}$. Consider a set $D = \varphi(a, \mathfrak{U})$, where $a \models p \in S_1$ and $D \subseteq p(\mathfrak{U})$. Assume D is a convex set of the form $[a, c)$. Then all such D's are of the form $[a, +\infty) \cap E(a, \mathfrak{U})$, for some equivalence relation E.

Now put a in the language and consider powers[2] of φ. [Something involving a limit of these and a notion of "φ-minimal element". It is something like: on ω when you pass to elementary extensions you have minimal elements of definable subsets].

(the example with three equivalence classes is (F) but not (LB)

[2] In the sense of composition of binary relations. Recall that it's everything inside $p(\mathfrak{U})$.

7.4 Itay Kaplan – On Kim-independence in NSOP1 theories

[slides][joint with Nick Ramsey and Saharon Shelah]

Definition 7.45. $\varphi(x, y)$ as SOP_1 iff there is ac ollection of tuples $\langle a_\eta \mid \eta \in 2^{<\omega}\rangle$ so that

- For all $\eta \in 2^\omega$ φ is consistent
- For all $\eta \in 2^{<\omega}$, if[...]

This is a more accessible definition

Definition 7.46. $\varphi(x, y)$ has an SOP_1 array iff ther eis a collequi of pairs $\langle c_i, d_i \mid i < \omega\rangle$ and some $k < \omega$ such that $\langle \varphi(x, c_i)\rangle$ is consistent, $\langle \varphi(x, d_i)\rangle$ is k-inconsistent, and $c_i \equiv_{c_{<i}, d_{<i}} d_i$ for all $i < \omega$.

Fact 7.47. T is NSOP_1 iff no formulas $\varphi(x, y)$ has an SOP_1-array.

Every simple theory is NSOP_1, every NSOP_1 is NTP_1.

SOP_1 was defined by Dzamonja and Shelah in 2004 (they were studying an order related to the Keisler order) and was later studied by Usvyatsov and Shelah, where a first example of a non-simple NSOP_1 was introduced (2008)

More recently, Chernikov and Ramsey provided (2016) more information. Tey proved a version of he Kim-Pillay characterisation for NSOP_1. Namely, if there is an independence relation satisfying certain properties, then the theory is NSOP_1. This characterisation can be used to provide many natural examples of NSOP_1 theories (and right now it seems like the only feasible way of proving that something is NSOP_1)

Theorem 7.48 (Chernikov-Ramsey)**.** Assueme there is an $\mathrm{Aut}(\mathfrak{U})$-invariant ternary $\mathop{\smile\!\!\!\!\!\!\mid}$ on small sets which satisfies

- Strong finite character: if $a \not\mathop{\smile\!\!\!\!\!\!\mid}_M b$, then there is $\varphi(x, b, m) \in \mathrm{tp}(a/bM)$ such that for any $a' \models \varphi(x, b, m)$, $a' \not\mathop{\smile\!\!\!\!\!\!\mid}_M b$
- Existence over models: if $M \models T$ then [...]
- Monotonicity
- Symmetry
- The independence theorem

(note the absence of Extension)

Example 7.49 (Ramsey)**.** The selector function: the model companion of the theory with two sorts F, O, where E is an eq rel on O, eva: $F \times O \to O$ is a function such that $\mathrm{eval}(f, o) E o$ and if $o_1 E o_2$ then $\mathrm{eval}(f, o_1) = \mathrm{eval}(f, o_2)$.

This should be the analogue for NSOP_1 theories of the random graph for simple theories: emblematic phenomena should be visible here.

Example 7.50 (Chernikov, Ramsey)**.** Parametrised simple theory (T simple and is a Fraïssé limit of a universal class of finite relational language with no algebraicity, then add a new sort for generic copies of models of T).

Example 7.51 (Chernikov, Ramsey)**.** ω-free PAC fields, i.e. PAC fields with Galois group $\hat{F}_\omega$, free profinite group with $\aleph_0$- generators. Was later extended to general Frobenius fields, essentially the same proof.

Example 7.52 (Kruckman, Ramsey)**.** If T is a model complete NSOP_1 theory eliminating $\exists^\infty$, then the generic expansion of T by arbitrary constant, function, and relation symbols is still NSOP_1 (exists by a theorem of WInkler, 1975)

Example 7.53 (Kruckman, Ramsey)**.** WIth the same assumption, T can be extended to a NSOP_1-theory with Skolem functions.

Example 7.54 (Chernikov, Ramsey)**.** vector spaces with a generic bilinear form (exist by Granger)

Example 7.55 (d'Elbée, ... (very new))**.** Algebraically closed field of positive characteristic with a generic additive subgroup.

It is natural to try and find an independence relation satisfying the criterion of Chernikov-Ramsey.

Definition 7.56. $\varphi(x, b)$ *Kim-divides* over M if there is a global M-invariant $q \supseteq \mathrm{tp}(b/M)$ such that $\{\varphi(x, b_i) \mid i < \omega\}$ is inconsistent when $\langle b_i \mid i < \omega \rangle$ is a Morley sequence in q over M.

[heuristic explanation: divides, and it can be witnessed by a Morely sequence in an invariant typ]

THis was suggested by Kim in Banff in 2009, and is related to Hrushovski's q-dividing and SHalh and Malliari's higher formula.

Lemma 7.57 (Kim's Lemma for Kim independence)**.** (NSOP_1) if φ Kim-divides and q is any global invariant type containing $\mathrm{tp}(b/M)$, then $\{\varphi(x, b_i)\}$ is inconsistent for any [...]

(it also implies NSOP_1) [more results] [those properties are not true for forking in NSOP_1 theories]

Theorem 7.58 (NSOP_1)**.** Kim-independence satisfies all the properties listed in the Chernikov-Ramsey criterion.

The main tool in the proofs of symmetry and independence theorem (the other are trivial) was tree Morley sequences. These are sequences which are indexed by an infinite tree. They play a similar role to Morley sequences in simple theories. Ca be defined of any height, but let's define the trees of height ω.

Definition 7.59. Let T_ω bet he set of functions : $[n,\omega) \to \omega$ with finite support. Put a tree order by $f \trianglelefteq g$ iff $f \subseteq q$. Let $f \wedge g = f \restriction m$ where $m = \min\{n < \omega \mid f \restriction [m,\omega) = g \restriction [m,\omega)\}$. The nth level is $P_n = \{f \mid \operatorname{dom} f = [n,\omega)\}$. Put a lexicographical order and denote with ζ_n the zero function with domain $[n,\infty)$.

This order has an infinite decreasing chain (it is ill-founded in the sense of set-theory, but not in the sense of descriptive set theory [???])

Definition 7.60. $\langle a_\eta \mid \eta \in T_\omega \rangle$ is a Morley tree over M iff

1. It is indiscernible with respect to $\{\trianglelefteq, <_{\mathrm{lex},\wedge,\leq_{\mathrm{len}}}\}$, where the last holds iff f is lower than g in the tree (the tree goes upwards and level P_0 is up wrt P_1).
2. For every $n < \omega$, there is some global invariant type q over M such that $\langle a_{\geq \zeta_{n+1} \frown \langle i \rangle} \mid i < \omega \rangle$ is a Morley sequence generated by q (note that it is a Morley sequence in a type in infinitely many variables, ordered in the sense of the sense of the tree).

Definition 7.61. A sequence $\langle a_n \mid n < \omega \rangle$ is Tree Morley sequence iff there is a tree as above suc that $a_n = a_{\zeta_n}$ for all n

Remark 7.62. To construct tree Morley sequences, one usually constructs a very tall tree, and then extract using Erdös-Rado.

Theorem 7.63 (NSOP_1)**.** If $\langle a_i, \mid i < \omega \rangle$ is a tree Morley sequence over M, then $\varphi(x, a_0)$ Kim-divides over M iff $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.

Theorem 7.64 (NSOP_1)**.** If $\langle a_i, \mid i < \omega \rangle$ is a universal witness for Kim-dividing, (if $\varphi(x, a_{<n})$ Kim-divides over M then the sequence of n-tuples from $\langle a_i \mid i < \omega \rangle$ witnesses this) then it is a Tree Morley sequence.

(there is a similar statement for Morley sequences in simple theories)

[Tree Morley sequences should coincide with Morley sequences in simple theories, if I understand correctly]

Theorem 7.65. Tree orley sequences exists: if $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ and $a \equiv_M b$ then there is a tree Morely sequence starting with a, b.

Corollary 7.66. Symmetry (for things with the same type).

(actually stuff is proven the other way around, symmetry is proven first than the last theorem)

Theorem 7.67. If $\langle a_i, \mid i < \omega \rangle$ is a Morley sequence over M, (i.e. indiscernible and forking-independent), then $\langle a_i, \mid i < \omega \rangle$ is a Tree Moreley sequence, and in particualr witnesses im-dividing

Theorem 7.68. TFAE for a NSOP_1 theory T

1. T is simple
2. $\mathop{\smile\!\!\!\!\!\!|}^K = \mathop{\smile\!\!\!\!\!\!|}^f$ over models
3. $\mathop{\smile\!\!\!\!\!\!|}^K$ satisfies base monotonicity over models.

In particular, this gets a new proof of a special case of the theorem of Shelah that $\mathsf{NTP}_2 + \mathsf{NTP}_1 =$ simple.

Corollary 7.69. If T is NSOP_1 and NTP_2, then T is simple

proof It is known that in NTP_2 if $\varphi(x, b)$ divides over M then it Kim-divides over M.

Remark 7.70. Transitivity fails: it is possible that $ab \mathop{\smile\!\!\!\!\!\!|}^K_M c$, $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$ but $a \not\mathop{\smile\!\!\!\!\!\!|}^K_M bc$. Funnily enough, if you replace [one of the two hypotheses, missed which one] with $\mathop{\smile\!\!\!\!\!\!|}^f$, then the "transitivity" works.

Theorem 7.71 (Kaplan, Ramsey (NSOP_1))**.** If $p \in S(M)$, then there is $N \prec M$ of size $\leq 2^{|T|}$ over which p does not Kim-fork.

It is not clear if this characterises NSOP_1. Anyway.
This was improved by Kaplan-Shelah:

Definition 7.72. For a set X, a family of C countable subsets of X is a club iff it is closed (countable union of elements from C is in C) and unbounded (every countable subset of X is contained in some member of C).

Consider the club filter and the associated notion of stationarity.

Fact 7.73. The club filter on X is generated by sets of the form C_F; where F is a collection of finitary functions on X and where C_F is the set of all countable $Y \subseteq X$ closed under members of F. The definition of club and the fact generalise to $[X]^\kappa$ instead of $[X]^\omega$.

Theorem 7.74. TFAE for a theory T

1. T is NSOP_1

2. If $p \in S(M)$ then for stationary many $N \prec M$ of size $|T|$, p does not Kim-divide over N
3. same with club many
4. if $p \in S(M)$, then there is a global extension q such that for are club many $N \prec M$ of size $|T|$, q does not Kim-fork over N.

(probably the cardinality of M can be bounded by something like $|T|^+$)

Remark 7.75. The proof of $1 \Rightarrow 3$ uses stationary logic, where one is allowed to add quantifiers of the form "for club many sets S, $\varphi(S)$ holds". (it is second-order; what is used from stationary logic is a version of upward Löwenheim-Skolem to blow up a stationary set where things do not divide to get a stationary set in the classical sense (as opposed to the definition with countable stuff) (say you have something of lenght ω_2, you have the "countable" thing, but you want a "real" stationary set...))

(actually the proof for one model, instead of club many, was even more complicated and invoked another black box)

7.5 Jana Maříková – Quantifier elimination for a certain class of convexly valued o-minimal fields

Model completeness was shown 4-5 years ago. We are going to see hoaw quantifier elimination follows from it.

Set up:

- R o-minimal field in $L \supseteq L_{\text{rings}}$ such that R has QE and is universally axiomatisable.
- V convex subring of R such that $k := V/\mathfrak{m}$ is o-minimal with the induced structure

(note: we are fixing a *complete* T satisfying the above)

Theorem 7.76. Suppose that $(R_0, V_0) \preceq (R, V)$. Then (R, V) has QE has in $L_{R_0} \cup \{V\}$ (old language plus constants for R_0 and V)

The proof uses the following.

Theorem 7.77 (Ealy, Maříková)**.** If $(R_0, V_0) \preceq (R, V)$, then (R, V) is model-complete in $L_{R_0} \cup \{V\}$.

Proof. Assume $(R', V') \equiv (R, V)$, $(R, V) \subseteq (R', V')$. Want: $(R; V) \prec (R', V')$. Use the following criterion:

Lemma 7.78. If $(R,V) \subseteq (R',V')$, then $(R,V) \preceq (R',V')$ if and only if there is no $a \in R'$ and no R-definable function f such that both a and $f(a)$ satisfy p, where $p(x)$ is the type of something $x > V$ and $x < R^{>V}$ and $a \in V'$ and $f(a) > V'$.

(this assumption is a kind of T-convexity: zoom in a type and definable functions are not allowed to move stuff from inside the valuation ring to outside of it)

The criterion gives model completeness. □

Lemma 7.79. If $(R',V') =\subseteq (R,V)$, then $(R',V') \preceq (R,V)$ (where (R,V) satisfies the hypotheses of the thing above) (assumptions: $R' \prec R$ and $V' = V \cap R'$.)

Proof. We have $(R_0,V_0) \subseteq (R',V') \subseteq (R,V)$. We claim that $(R_0,V_0) \preceq (R',V')$; otherwise, let f be given by the criterion above; use o-minimality to get an interval with endpoints in [something] such that f is either strictly increasing or strictly decreasing in the interval and prove that f is strictly increasing. Since $f(V_0) \subseteq V_0$ we have $f(V) \subseteq V$. S $f(V') \subseteq V'$, which is a contradiction.

Now we have $(R_0,V_0) \preceq (R',V') \subseteq (R,V)$. So $\mathrm{Th}(R_0,V_0) = \mathrm{Th}(R',V')$. But then $\mathrm{Th}(R',V') = \mathrm{Th}(R,V)$, and by model completeness the inclusion is elementary. □

Corollary 7.80. $\mathrm{Th}(R,V)$ has a universal axiomatisation.

Proof. It is implied by its universal theory. □

Remark 7.81. For a model-complete theory, having quantifier elimination is equivalent to $T^\forall$ having the amalgamation property.

Proof of Theorem 7.76. Since $T^\forall \vdash T$, we have the amalgamation property. □

Corollary 7.82. (R,V) has definable Skolem functions

Proof. Quantifier elimination together with a universal axiomatisation always get you definable Skolem functions. □

7.6 Daniel Palacín – A property of pseudofinite groups

[slides][joint with Nadja Hempel]

Recall that a pseudofinite group is an infinite model of the first-order theory of the theory of finite groups. Equivalently, it is elementarily equivalent to a non- principale utlraproduct of finite groups.

SImilarly, one can define pseudofinite fields.

Example 7.83. Torsion free divisible abelian groups, infinite extraspecial p-groups of odd expoennet p, general linear groups over a pseudofinite fields,...

Example 7.84. Non-examples are $(\mathbb{Z}, +)$, free groups, infinite dihedral subgroup, Higman group... For instance you can show this by finding injective, non-surjective definable maps. For the Higman group, you use that it is finitely presented but has no finite quotient.

We want to study what impact have centralisers on the structure of pseudofinite groups

Theorem 7.85 (Burnside)**.** Any finite gorup admitting a fixed-point-free[3] involutory automorphism is abelian.

(the idea is that it would be the inverse map)

Theorem 7.86 (Brauer and Fowler, 1955)**.** There are only a finite number of finite simple groups with a given centraliser of an involution.

Theorem 7.87 (Hempel, Palacín)**.** Any pseudofinite group has an infinite abelian subgroup.

The main ingredients are: the Feit-Thompson Theorem: a finite group without an involution is solvable, and standard things from infinite group theory.

Definition 7.88. The FC-center of a group G is defined as $\mathrm{FC}(G) = \{x \in G \mid [G : C_G(x)]\text{is finite}\}$

Equivalently, it is the set of elements with finite x^G.

It is a characteristic subgroup and its derived subgroup is a periodic subgroup. It is a union of definable groups.

If there is a natural number k such that $\mathrm{FC}(G) = \{x \in G \mid [G : C : G(x)] \leq k\}$, then $\mathrm{FC}(G)'$ is finite, i.e. $\mathrm{FC}(G)$ is finite-by-abelian.

Example 7.89 ($G = \mathrm{FC}(G)$)**.** Infinite direct sum of[...]

Lemma 7.90. If G is a pseudofinite group containing an involution i with a finite centraliser, then either $C_G(i) \cap \mathrm{FC}(G)$ contains an involution or $G = C_G(i) \cdot \mathrm{FC}(G)$.

Proof idea. Set $I(x)$ to denote the set of involutions in $C_G(x)$, and for any $g \in G$, let X_g be the finite set

$$I(i) \cup I(i^g) \cup g^{-1}C_G(i) \cup C_G(i)g^{C_G(i)}$$

Using the fact that any two involutions are either conjugate or have an involution centralising both of them, show that fr any $j \in G$ we have $X_g \cap X_g^h \neq \emptyset$. Thus $G = \bigcup_{a,b \in X_g} \{u \in G \mid a^u = b\}$. Note that this union is finite. Then $X_g \cap \mathrm{FC}(G) \neq \emptyset$ by a lemma of B.H. Neumann. Note: either $I(i) \cap \mathrm{FC}(G) \neq \emptyset$ or $g^{-1}C_G(i) \cap \mathrm{FC}(G) \neq \emptyset$. □

[3]Of course, except for the identity.

Now to prove the Theorem: first show that a pseudofinite group G has a non-trivial element with an infinite centraliser:

- Suppose not
- By the Lemma, G has no involution
- By Feit-Thompson, in any finite F without involutions there is a non-trivial element x that commutes with all its conjugates, i.e. $\langle x^F \rangle$ is abelian.
- As G is pseudofinite, there is some non-trivial $h \in G$ such that $h^G \subseteq C_G(h)$, yielding that $G/C_G(h)$ is finite and so is G, a contradiction.

Without loss of generality, assume G is periodic (otherwise, take an element of infinite order, then take the center of the centraliser).

- Let x_0 be an element of G with infinite centraliser
- Consider the pseudofinite $G_0 = C_G(x_0)/\langle x_0 \rangle$ and applyu the first step to find some x_1 in $C_G(x_0$ whose class $\bar{x}_1$ in G_0 is non-trivial and has an infinite centraliser
- [missed the rest]

Remark 7.91. It is reasonable to use Feit-Thompson since it is also needed to show the existence of an infintie abelian subgroup ina locally finite group (Hall, Kulatilaka)

Remark 7.92. The proof is surprisingly easy compared to the existence of an infinite albelian subgroup in any profinte group!

At the level of finite groups one immediately obtains (maybe it is known or can otherwise be shown directly)

Corollary 7.93. *For each n, there are only finitely many finite groups in which the centraliser of every non-trivial element has size at most n.*

Assuming a uniform chain condition oncentralisers (up to bounded index) one easiluy obtains an infinite definable (finite-by)-abelian subgroup like in stable (simple) theories. Consequently, every pseudofinite group of thron-rank one is finite-by-abelian-by-finite (Wagner 2015).

Theorem 7.94 (Wagner 2015)**.** *There is a function $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that fro any $n, m \in \mathbb{N}$ if G is a finite group in which one cannot finite elements $a_0, \ldots, a_n$ satisfyuing $[C_G(a_0, \ldots, a_{i-1}) : C_G(a_0, \ldots, a_i)] \geq n$ for every $i \leq m$, then there is a subgroup H of G such that $|H'| \leq f(n,m)$ and $|G| \leq f(n,m)|H|^{f(n,m)}$.*

Definition 7.95. A group has restricted centralisers iff the centralisers of any element is finite or has finite index.

Example 7.96. FC-groups, infinite dihedral group, Tarski monsters, an group of thorn-rank one.

Theorem 7.97 (Shalev, 1994)**.** A profinite group having restricted centralisers is finite-by-abelian-by-finite.

In the proof Shalev uses the solution to the restricted Burnside Problem for compact groups, and a result on finite groups deu to Hartely-Meixner an Khukro (depends on the classification of finite simple groups)

Theorem 7.98 (Hartley and Meixner 1981, Khukhro 1993)**.** There are $f\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ and $h\colon \mathbb{N}\to\mathbb{N}$ such that given a finite group G admitting an automorphism of prime order p with centraliser of order n, there is a nilpotent subgroup of index bounded by $f(n,p)$ and whose nilpotency class is at most $h(p)$.

Theorem 7.99 (Hartley and Mexiner, 1980)**.** There is a function $f\colon \mathbb{N}\to\mathbb{N}$ such that if G is a finite roup admitting an involutory automorphisms α with $|C_G(\alpha)\le n|$, then there is a normal H in G such that $[G:H]\le f(n)$ and $H'\le C_G(\alpha)$.

Using the Main Lemma Palacin and Hempel obtained a model-theoretic proof of this. The interesting thing would be having such a proof for Theorem 7.98

Theorem 7.100 (Hempel, Palacin)**.** The FC-center of any psudofinite group with resrticted centralisers is defianble and has finite index.

The proof uses Theorem 7.98. If the pseudofinite group is $\aleph_0$-saturated, then the FC-center is finite-by-abelian.

Corollary 7.101 (Hempel, Palacin)**.** There is a function $f\colon \mathbb{N}\to\mathbb{N}$ such that if G is a finite group such that for all x the size of $C_G(x)$ or $G/C_G(x)$ is at most n, ten there is a characteristic subgroup H of G such that the size of G/H and H' are bounded by $f(n)$.

Chapter 8

Friday 7 July

8.1 Jamshid Derakhshan – Definable sets, Euler products of p-adic integrals, and zeta functions

[slides] [basically a paper; look directly at those]

8.2 Pierre Simon – Finitely generated dense subgroups of Aut(M)

[joint with Itay Kaplan]

Recall that S_∞ is a topological group with the product/pointwise convergence topology, and that closed subgroups are precisely the ones of the form $\mathrm{Aut}(M)$.

Generally, you cannot reconstruct M from $\mathrm{Aut}(M)$: for instance a lot of M are rigid. Anyway, you *can* do this for ω-categorical structures, up to biinterpretability.

ω-categoricity of M translates to $\mathrm{Aut}(M)$ being oligomorphic, i.e. having, for every n, finitely many orbits on M^n.

Let $G = \mathrm{Aut}(M)$, and let $H \leq G$ be a closed subgroup. This corresponds to some M', where M' is an expansion of M.

Example 8.1. Some examples of ω-categorical structures:

- $(\mathbb{N}, =)$ (here $G = S_\infty$)
- DLO
- the countable dense circular order
- Fraïssé limit of trees (in the sense of: poset where every set of predecessors is linearly ordered)

Question 8.2. Let Γ be a finitely generated group. Does Γ have a faithful oligomorphic action on $\mathbb{N}$? Equivalently, is there an ω-categorical structure M such that $\Gamma \leq \operatorname{Aut}(M)$ and Γ is dense?

Question 8.3. Given an ω-categorical M; what are its finitely generated/countable dense subgroups of $\operatorname{Aut}(M)$?

Question 8.4. Is there a finitely generated dense subgroup?

Example 8.5. $S_\infty = \operatorname{Aut}(\mathbb{Z})$ can be generated[1] by a shift $n \mapsto n+1$ and a 2-cycle (12).

Finitely generated subgroups have something to do with compact quotients. Given an ω-categorical M, we can consider $K := \operatorname{Aut}(\operatorname{acl}^{\mathrm{eq}}(\emptyset))$. This is a profinite group, and $G = \operatorname{Aut}(M)$ projects naturally $G \twoheadrightarrow K$ (the map is continuous).

Fact 8.6. Any profinite group arises in such a way. Moreover, K is the maximal compact quotient of G.

Remark 8.7. If finitely generated/dense subgroups exist in G, then so they do in K: just take the projection.

So we have an exact sequence

$$1 \to G^0 \to G \to K \to 1$$

How do we get a 2-generated subgroup? We will see a sufficient condition.

Definition 8.8. If $A, B \subseteq M$, we say that $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ns}} B$ (A is *non-splitting independent from* B) iff for all $a, a' \in A$, $b, b' \in B$ such that $a \equiv a'$ and $b \equiv b'$ we have $ab \equiv a'b'$.

Definition 8.9. $\sigma \in \operatorname{Aut}(M)$ is *repulsive* iff for any finite $A \subseteq M$ there is n such that $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ns}} \sigma^n(A)$.

Example 8.10. In DLO, with $\mathbb{Q}$ as a model, the shift $x \mapsto x+1$ is repulsive.

Proposition 8.11. If σ is repulsive, then there is $f \in \operatorname{Aut}(M)$ such that $\{\sigma^n f \sigma^{-n} \mid n \in \mathbb{N}\}$ is dense.

Proof Idea. Build f by back and forth/induction. Given $f_0 \colon c \mapsto d$, consider $\bigcup_{a,b} ca \mapsto b$. Pick N such that $cd \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ns}} \sigma^n(ab)$ and take $f_0 \cup \{\sigma^n(a) \mapsto \sigma^n(b)\}$. ∎

Definition 8.12. An independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ is *canonical*[2] iff it satisfies

[1]Keep in mind this means it is generated *topologically*
[2]Probably not a good name.

- Existence, extension, subsets (maybe not the strongest forms, but it's not the most important thing here)
- Stationarity over $\emptyset$: if $a \perp_{\emptyset} b$ and $a' \perp_{\emptyset} b'$, and $a \equiv a'$, $b \equiv b'$, then $ab \equiv a'b'$.
- Transitivity: (if $A \perp_{DC} B$ and $D \perp_C B$, then $AD \perp_C B$) and (if$A \perp_C D$ then $A \perp_C BD$).

Remark 8.13. Stationarity says in particular implies that $\perp \Rightarrow \perp^{\mathrm{ns}}$, but it also tells you you can choose the extension.

Remark 8.14. We do now assume symmetry.

Example 8.15. If T is stable and $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$, then you can take $\perp$ to be non-forking

Example 8.16. If DLO, you can take $A \perp_{\emptyset} B$ if $A < B$.

Example 8.17. In the random graph, you can take $A \perp_C B$ iff $[A \cap B \subseteq C?]$ and there are no edges between $A \setminus C$ and $B \setminus C$.

Proposition 8.18. If M admits a canonical independence relation, then $\mathrm{Aut}(M)$ has a repulsive automorphism.

Proof. The idea of the proof is to mimic the shift we saw before. This is done by starting with M_0, taking a $\perp$-independent copy M_1, and amalgamate them in some M_{01}. Do this for every finite sequence of positive naturals, and then take as σ the automorphism induced by shifting the indices by 1. □

Remark 8.19. Stationarity implies that $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$.

An example where this does not work is a circular order: you cannot even get a, b, c, d such that $ab \not\perp^{\mathrm{ns}} cd$. If a, b, c, d are arranged in that order, $ab \equiv ba$, but still $ab \not\perp_{cd} ba$. But if M_a is obtained by naming one point, then $\mathrm{Aut}(M_a) \cong \mathrm{Aut}(\mathsf{DLO})$. How do we show that this is dense? If $\sigma\colon a \mapsto a'$, then $\mathrm{Aut}(M) = \overline{\mathrm{eqAut}(M_a), \sigma}$. [missed the argument for this]

This idea can be generalised a little bit.

Suppose N is an expansion of M, where you name an M-definable set. So something defined originally with parameters A now becomes $\emptyset$-definable. Find $\sigma\colon A \mapsto A'$ such that $\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(A') = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Then you can argue as in the circular order example.

Theorem 8.20. Let M be ω categorical and suppose it has an expansion N which is still ω-categorical. Assume that $\mathrm{Aut}(M)$ has no compact quotient[3] Then there are $f, g \in \mathrm{Aut}(M)$ such that $\mathrm{Aut}(M) = \overline{\langle \mathrm{Aut}(N), f, g \rangle}$.

[3] For simplicity. Actually it can be relaxed.

Question 8.21. Does every finitely homogenous[4] ω-categorical M have an ω-categorical expansion with canonical $\mathop{\smile\!\!\!\!\!\!|}\,$?

The only examples know to the speaker where you need an expansion are the circular order and the unrooted tree.

Theorem 8.22 (KPT)**.** *M has the Ramsey property if and only if $\mathrm{Aut}(M)$ is extremly amenable, i.e. every compact action $\mathrm{Aut}(M) \curvearrowright K$ has fixed point.*

Proposition 8.23. *If M (ω-categorical) has a repulsive automorphism σ, then for all actions $\mathrm{Aut}(M) \curvearrowright K$, for K metrisable compact, and for all $x \in K$, there is a σ_* conjugate to σ such that $\overline{\{\sigma_*^n(x) \mid n < \omega\}}$ contains a subflow.*

In particular if M is Ramsey, that will contain a fixed point.
[some open questions]

Question 8.24. Any other consequences of having such an independence relation?

[back and forth constructions usually give you free groups: it's very easy to kill relations]

8.3 Daniel Hoffmann – Model theoretic dynamics in a Galois fashion

[slides] [personal interpretation of the universe]
The point is trying to attack the NSOP part of the universe from the stable part.
Examples of simple theories: ACFA, $\mathbb{Q}$ACFA, PAC fields,[...]
Doubt: are simple theories studied enough?

Question 8.25. Is ACFA a generic simple theory, in some sense?

We will say "ACFA-like" to mean a stable theory enriched with an automorphism.
If we remove "fields" from ACFA we get something that was studied by Chatzidakis and Pillay in 1998 in *Generic and simple theories.*
[G-TCF] [literature; illegible]
Fix L, G and let $L^G)L \cup \{\sigma_g\}_{g \in G}$. Let T eliminate quantifiers and imaginaries, and assume it is stable and PAPA (a weakening of being a model companion). Let T_0 be the theory of models of T with σ, and let T_A be the model companion of T_0. [this has some problems]
Instead, let T be an L-theory, and assume T has a model companion T^{mc} which is stable and eliminates quantifiers and imaginaries. Then let T_G be the theory of models of T with G-automorphisms and produce T_G^{mc} [???]

[4] Otherwise is not true.

Fact 8.26. If T^{mc} is stable, then T^{mc}_G is simple

A long term plan is trying to prove the following:

- If T' is simple (probably more restrictions), then there are L, T and G such that T^{mc}_G exists and is bi-interpretable with T'
- Geometric description of simple theories n the style of ACFA

Assume T^{mc} and T^{mc}_G exist and T^{mc} is stable and eliminates quantifiers and imaginaries.

Fix a monster $(\mathfrak{U}, (\sigma_g)_{g\in G}) \models T^{\mathrm{mc}}_G$ and a $|\mathfrak{U}|^+$-saturated monster $\mathfrak{D} \models T^{\mathrm{mc}}$ such that $\mathfrak{U} \subseteq \mathfrak{D}$.

After use of Galois theory, we can say more about hwo $\mathfrak{U}$ is contained in $\mathfrak{D}$. E.g. $\mathrm{acl}^{\mathfrak{U}}_{L^G}(A) = \mathrm{acl}^{\mathfrak{U}}_L(G \cdot A) = \mathrm{acl}^{\mathfrak{D}}_L(G \cdot A) \cap \mathfrak{U}$.

Why QE and EI? Because of this

Corollary 8.27. Assume $E, A, B \subseteq \mathfrak{D}$, A regular over E, i.e. $E \subseteq A$ and $\mathrm{dcl}^{\mathfrak{D}}_L(A) \cap \mathrm{acl}^{\mathfrak{D}}_L(E) = \mathrm{dcl}^{\mathfrak{D}}_L(E)$ (it's actually a condition about $\otimes$). Assume that $f_1, f_2 \in \mathrm{Aut}_L(\mathfrak{D})$, and they adree on E. If $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathfrak{D}}_E B$ and $f_1(A) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathfrak{D}}_{f_1(E)} f_2(B)$, then there is $h \in \mathrm{Aut}_L(D)$ suc that $h \restriction_a = f_\restriction A$ and [something else]

Definition 8.28. $A \overset{\circ}{\mathop{\smile\!\!\!\!\!\!\mid}}_E B$ means $G \cdot A \mathop{\smile\!\!\!\!\!\!\mid}_{G\cdot E} G \cdot \mathop{\smile\!\!\!\!\!\!\mid} B^{\mathfrak{D}}$.

Definition 8.29. Say that the algebraic closure split in $\mathfrak{D}$ over $\mathfrak{U}$ if for all small $(M, \bar\sigma) \prec (\mathfrak{U}, \bar{s}igma)$ and all small $A \subseteq \mathfrak{U}$ such that $M \subseteq A$, it follows that $\mathrm{acl}^{\mathfrak{D}}_L(A) = \mathrm{dcl}^{\mathfrak{D}}_L(\mathrm{acl}^{\mathfrak{D}}_L(A) \cap \mathfrak{U}, \mathrm{acl}^{\mathfrak{D}}_L(M))$.

Fact 8.30. Of for $\mathfrak{U} = \mathfrak{D}$ (e.g. $G = \mathbb{Z}$, like ACFA). Ok in some more cases]

Theorem 8.31. If algebraic closures split, then $\overset{\circ}{\mathop{\smile\!\!\!\!\!\!\mid}}$ satisfies theindependence over a model theorem (T^{mc}_G is simple and $\overset{\circ}{\mathop{\smile\!\!\!\!\!\!\mid}}$ is forking indepdence.

Fact 8.32. No hope for: stability of T^{mc}_G in general (ACFA). No hope for superstable implies supersimple ($\mathbb{Q}$ACFA). No hope for removing "algebraic closures split" (if not, then it is not bounded, e.g. D_∞ − TCF exists and is not simple, because models are not bounded.

Other results: T^{mc}_G has semi-quantifier elimination (similar as in ACFA). Also, it codes finite tuples. Moreover, if $\overset{\circ}{\mathop{\smile\!\!\!\!\!\!\mid}}$ is forking independence, then it has geometric elimination of imaginaries (each tuple is interalgebraic with a real tuple). Anyway it does not necessarily have weak elimination of imaginaries (CCMA).

8.4 Ningyuan Yao – On definable f-generics in distal NIP theories

[examples of unstable simple and unstable NIP theories][definition of NIP with indiscernible sequences][Th($\mathbb{Q}_p$) (and Presburger?) are distal]

1-dimensional definable subsets in o-minimal structure is defined by order and paramters. In $\mathbb{Q}_p$ is defined by Boolean combinations of $v(x-a) > m$ and $C([\ldots])$.

[usual notations, generic types in definable groups]

Theorem 8.33 (Fundamental theorem of stable groups)**.** If T is stable and $\mathcal{P}_G$ is the space of global generic types, then $\mathcal{P}_G \neq \emptyset$, G^{00} xists and $\mathcal{P}_G$ is homeomorphic to G/G^{00}.

How to generalise to unstable NIP? $\mathfrak{P}_G$ may not exist.

Definition 8.34. p is f-generic iff for any $g \in G$ and every L_{M_0}-formula $\varphi \in p$, $g\varphi$ does not *divide* over M_0

[more definitions] [G^{00} exists in NIP theories]

The space of f-generic, or weakly generic types is not homeomorphic to G/G^{00}. So we meed more invariants. [topological dynamics, Newelski] [recall weakly generics are the closure of the almost periodic types]

Theorem 8.35 (Chernikov-Simon)**.** If G is NIP definably amenable, then [that thing] and weakly generic types are the same as f-generic types, same as types with bounded orbit, same as G^{00}-invariant types.

[more stuff]

Theorem 8.36 (Conversano-Pillay)**.** Assuming that T is an o-minimal expansion of RCF, a definable group G is definably amenable iff if fits into a

$$0 \to H \to G \to K \to 0$$

where H is definable torsion-free and K is defianbly compact.

The parts H and K are in a sense orthogonal.

The torsion free part H has a global f-generic type which is $\emptyset$-definable. $H^{00}?H$, so every f-generic type is almost periodic.

The compact part K has fsg. Generic types exist and every f-generic type is almost periodic (Newelski).

Definable global types and finitely satisfiable global type commute, so are orthogonal (Simon).

[fsg]

Theorem 8.37 (HPP)**.** In RCF, G has fsg iff it is definably compact.

Theorem 8.38 (Onshuus-Pillay)**.** Same in $\mathbb{Q}_p$.

Definition 8.39. dfg: global type p such that every left translate of p is $\emptyset$-definable (p has bounded orbit)

In RCF, G has dfg if and only if G is torsion-free. What about $\mathbb{Q}_p$?

Question 8.40. Would dfg groups be suitable analogs of torsion free groups defined in o-minimal theories in other NIP theories?

Theorem 8.41 (Pillay, Yao)**.** Assume NIP. If $p \in S_G(M)$ is dfg, then the orbit of p is closed, so p is almost periodic, and $G^{00} = G^0$.

Question 8.42. Assuming distality and NIP, suppose that G has dfg. Is every f-generic type almost periodic?

The answer is positive (trivially) in o-minimal expansions of RCF. Consider Presburger Arithmetic. Let G_a be the additive group and $G = G_a^n$.

Theorem 8.43 (Conant, Vojdani)**.** [missed it] [among the other things, dfg's are almost periodic]

Consider $\mathbb{Q}_p$.

- every f generic type of G_a^n and G_m^n is $\emptyset$-definable
- [more stuff]

Let $\mathfrak{U} \equiv \mathbb{Q}_p$. Let UT_n be the subgroup of upper triangle matrices.

Theorem 8.44 (Pillay, Yao)**.** $\mathrm{tp}(\alpha/\mathfrak{U}) \in S_{UT_n}(M)$ is f-generic if and only if [characterisation]

[more positive examples where under dfg every f-generic type is almost periodic]

8.5 Lou van den Dries – Hardy fields and transseries: a progress report on ongoing joint work with Matthias Aschenbrenner and Joris van der Hoeven.

[slides]

8.5.1 Orders of Infinity and Hausdorff Fields

This originates with Du Bois-Reymond, a rival of Cantor (developed his own diagonal method).

Work was picked up by Hausorff and Hardy. [names and history]

Du Boid-Reymond: let $\mathcal{C}$ be the ring of germs at $+\infty$ of real alued functions with domain a subset of $\mathbb{R}$ including some $(a, +\infty)$ on which the function is continuous. Define $f \preceq g$ iff $|f(t)| \leq c|g(t)|$ eventually.

This is a preorder and yields an equivalence relation denoted $\asymp$. Clearly, $\preceq$ passes to the quotient. The equivalence classes are the "orders of infinity" of Du Bois-Reymond.

Remark 8.45. This is not a linear ordering in general.

We will speak of functions, but remember that actually their are germs.

Example 8.46. $x \asymp 2x$, but $x \not\prec 2x$.

Definition 8.47. A *Hausorff field* is a subfield of $\mathcal{C}$.

Example 8.48. $\mathbb{Q}, \mathbb{R}, \mathbb{R}(x), \mathbb{R}(\sqrt{x}), \mathbb{R}(x, e^x, \log x)$

For any expansion $\tilde{\mathbb{R}}$ of the field of reals TFAE:

- $\tilde{\mathbb{R}}$ is o-minimal
- The germs at $+\infty$ of its definable functions $\mathbb{R} \to \mathbb{R}$ form a Hausdorff field (and indeed, there is a natural way to make it into an elementary extension of $\tilde{\mathbb{R}}$)

Let H be a Hausdorff field. For every nonzero $g \in H$ there is $h \in H$ with $gh = 1$, so $g(t) \neq 0$ eventually, and thus $g(t) > 0$, eventually, or $g(t) < 0$ eventually. Thus J is naturally a totally ordered field, ad the set of orders of infinity in H is totally ordered.

Fact 8.49. *H has a unique algebraic Hausdorff field extension that is real closed.*

Hausdorff was interested in "maximal" objects and their order type. Every Hausdorff field is contained in a maximal Hasu field (by Zorn[5]). Maximal Hausdorff fields are RCF by the above.

Hausorff worked with a variant: sequences instead of functions, but it is trivial to translate between the two objects.

Maximal Hausorff fields have uncountable cofinality:

Fact 8.50. *THe underlyuing ordered set of a maximal Hausorff field H is an η_1-set: if $A, B \subseteq H$ are countable and $A < B$, then $A < h < B$ for some $h \in H$.*

[5]At that time Zorn's lemma did not exist. And actually Zorn's Lemma is by Hausdorff. It was called Hausdorff Maximality Principle

Corollary 8.51 (CH)**.** All maixmal Hausdorff fields are isomorphic.

Question 8.52. Is this true without CH?

8.5.2 Hardy Fields

A Hardy field is a Hausdorff field whose germs can be differentiated, with the resulting derivation as an extra primitive. This leads to a much richer theory and an intriguing open problem:

Question 8.53. What can we say about maximal Hardy fields?

Consider $\mathcal{C}^1$ (in the sense of "eventually continuously differentiable") and note that it is a subring of $\mathcal{C}$. Note that $f' \in \mathcal{C}$, but not necessarily in $\mathcal{C}^1$.

Definition 8.54. A Hardy field is a subfield of $\mathcal{C}^1$ that is closed under $f \mapsto f'$.

Hardy fields are thus not only orederd fields, but also differential fields. For f in a Hardy field we have either $f' > 0$ or $f' = 0$ or $f' < 0$, so f is either eventually strictly increasing, constant or strictly decreasing (this is false in general for f in a Hausdorff field).

Remark 8.55. Even if a Hardy field is closed under derivation, one cannot in general choose a C^∞ representative for the germ (you may have to take a smaller interval to get more derivatives).

Let H be a Hardy fields. The ordering and derivation interact: if $f \in H$ and $f > n$ for all n, then $f' > 0$. Asymptotic relations in H can be "differentiated and integrated": $f \preceq g \iff f' \preceq g'$, for nonzero $f, g \not\asymp 1$ ($\asymp 1$ means both bounded and bounded away from 0).

Remark 8.56. All the example of Hausdorff fields mentioned before are Hardy fields.

Theorem 8.57 (Hardy's theorem)**.** COnsidere the functions $f(x)$ that can be expressed in terms of x using real constants, $+, \times, 1/-, \exp, \log$ and which are defined for all sufficiently large real arguments Their germs form a Hardy field

Main thing to show: such a function is eventually constant, [positive or negative?]

This theorem was hard to prove. Now can be seen as a byproduct of some basic extension results on Hardy fields H:

- H has a unique algebraic Hardy field extension that is real closed
- If $h \in H$, then e^h generates a Hardy field $H(e^h)$

- any antiderivative $g = \int H$ of any $h \in H$ generates a Hardy field $H(g)$

Special cases of the last item: $H(\mathbb{R})$ and $J(x)$ are Hardy fields, and if $h \in H^{>0}$, then $H(\log h)$ is a Hardy field. Thus maximal Hardy fields contain $\mathbb{R}$, are real closed, and closed under exponentiation and integration.

Remark 8.58. Not any continuous function can live in an Hausdorff field: e.g. $\sin x$.

Also, for o-minimal $\tilde{\mathbb{R}}$, the germs at infinity form a Hardy field.
Work in progress (ADH) has amongits goals to prove for Hardy fields H:

1. for any differential polynomial $P(H) \in H[Y, Y', Y'', \ldots]$ and $f < g$ in H with $P(f) < 0 < P(g)$ there is a y in a Hardy field extension of H such that $f < y < g$ and $P(y) = 0$. This is a kind of intermediate valued property for differential polynomials (this has been proven by van den Dries in the case of polynomials of order 1)

2. For any countable sets $A < B$ in H there is a y ina Hardy field extension of Hsuch that $A < y < B$ (which clearly implies that maximal Hardy fields are η_1)

The first conjecture implies that all maximal Hardy fields are elementarily equivalent. Both conjectures together imply that under CH all maximal Hardy fields are isomorphic. (Remark: Hardy fields cannot be $\aleph_1$-saturated, because the field of constants is the reals and is definable, so the isomorphism is true for other reasons).

8.5.3 Transseries

[the book about model theory of transseries came out. 75 dollars]

Hardy: the logarithmic-exponential functions seem to cover all orders of infinity that occur naturally in mathematics. He also suspected that the order of infinity of the compositional inverse of $(\log x)(\log\log x)$ differs fromt hat of any logarithmic-expoenntial function. He was right.

For a more reavealing view of orders of infinity and a more comprehensive theory we need transseries. For example, you can use to prove Hardy's suspicion.

Example of a transseries:

$$e^{e^x + e^{\frac{x}{2}} + e^{\frac{x}{4}} \cdots} - 3e^{x^2} + 5x^{\sqrt{2}} - (\log x) + 1 + x^{-1} + x^{-2} + \ldots + e^{-x}$$

In a transseries, thing of x as positive infinite $x > \mathbb{R}$. The monomials are transmonomials.

The definition is lengthy. For each transseries there is a finite bound on the nesting of exp and log in its transmonomial. So $\mathbb{T}$ is not spherically complete.

The construction of $\mathbb{T}$ involves building an isomorphism $f \mapsto e^f$ of the ordered additive gorup of $\mathbb{T}$ onto its multiplicative group $\mathbb{T}^>$ of positive elements, wit inverse $g \mapsto \log g$. The sequence x, e^x, e^{e^x}, etc is cofinal in $\mathbb{T}$ and x, $\log x$, $\log\log x$, etc is coinitial in $\mathbb{T}^{>\mathbb{R}}$ (they are called log-*atomic* elements.

As an ordered exponential field, $\mathbb{T}$ is an elementary xtension of the ordered exponential field of real numbers.

Every $f \in \mathbb{T}$ can be differentiated term by term. [example] We obtain a defivation $f \mapsto f'$ on $\mathbb{T}$. Its constant field is $\mathbb{R}$.

Every $f \in \mathbb{T}$ has an antiderivative in $\mathbb{T}$ (which, considered as series, may diverge, but we consider them as formal series)

for $f, g \in T$ we can define $\preceq, \asymp, \prec$ in the same way. $f \preceq g$ is equivalent to the leading transmonomial being smaller. [example]

As in Hardy fields, $f > \mathbb{R} \Rightarrow f' > 0$ and we cna differentiate and integrate dominance.

We shall consider $\mathbb{T}$ as a valued ordered differntial field. Model theoretically, the language is $L_{\text{rings}} \cup \{\partial, \leq, \preceq\}$. More generally, let K be any ordered differential field with constant field C. This yields a dominance relation defined in the same way, and we view K accordignly s an L-structure. Introduce the valuation ring as $\mathcal{O} = \{f \in K \mid f \preceq 1\}$, which is the convex hull of C in K, and define the maxiaml ideal o accordingly.

Definition 8.59. An H-field is an ordered differential field K such that

1. $f > C \Rightarrow f' > 0$
2. $\mathcal{O} = C + o$

Example 8.60. Hardy fields that contain $\mathbb{R}$. Differential subfields of $\mathbb{T}$ that contain $\mathbb{R}$.

In particular, $\mathbb{T}$ is an H-field, but $\mathbb{T}$ has further basic elementary properties that do not follow from this: its derivation is small ($f \prec 1 \Rightarrow f' \prec 1$), and it is Liouville closed (real closed and for every f there are g, h such that $g' = f$ and $h \neq 0$ and $h'/h = f$)

20 years ago, the authors conjectured that the theory of H-fields has a model companion, and that's the theory of transseries. This has now been proved.

Definition 8.61. An H-field K has IVP (intermediate value proprety) iff for every $P(Y) \in K[Y, Y', \ldots]$ and all $f < g$ in K with $P(f) < 0 < P(g)$ there is $y \in K$ with$f < y < h$ and $P(y) = 0$.

Theorem 8.62. $\mathbb{T}$ is completely axiomatisd byt he requirements:

- being an H-field with small derivation

- being Liouville closed
- having IVP

(note: in the book IVP is not present, they used something more complicated; IVP was introduced more recently)

IVP is an afterthough. By it explains why maximal Hardy fields, modulo that conjecture, are elementary equivalent.

All H-fields embed into Lioville closed H fields having IVP, and the latter are exactly the existentially closed H-fields. Thi

Theorem 8.63. The theory of Lioville closed H-fields having IVP is model complete (and is the model companion of...)

Byproduct of the proof: if K is a Lioville closed H-field wit IVP, then K has no proper differntial-algebraic H-field extension with the sameconstant field.

IVP refrs to te ordering, bat the valuation given by $\preceq$ is more robust and useful.

IVP comes from two more fundamental property: ω-freeness and newtonianity. This makes sense for more general differnetial field satisfying that you can differentiate and integrate $\preceq$ (withthose proviso)

To give an idea of these (technical) notions, let $P(Y) \in K[Y, Y', \ldots]$ be a nonzero differential polynomial. We wish to understandhow the function $y \mapsto P(y)$ behaves for $y \in K$ not too far from the constant field C.

It turns out that this function only reveals its true colors after rewrtiting P in terms of a derifation$\varphi\delta$ with $\varphi \in K^\times$ sufficiently large wrt $\preceq$ subject to $\varphi\delta$ being small. This rewritten P will eventually be of the form $a \cdot (N(Y) + R(Y))$ with $a \in K^\times$, nonzero $N(Y) \in C[Y, Y', \ldots,]$ independent of φ, and where the coefficients of R are infinitesimals (in o). We call $N(Y)$ the *Newton polynomial* of P.

K is ω-free iff for all P as above its Newton polynomial has the form $A(Y)(Y')^n$ with $A(Y) \in C[Y]$. This is equivalent to a lot of other conditions.

K is newtonian iff for all P as above with Newton polynomial of degree 1 we have $P(y) = 0$ for some $y \preceq 1$.

For H-fields IVP implies ω-free and newtonian. The converse holds for Lioville closed H-fields, but it is known to fail in general.

Back to Hardy fields, from the second part of the talk it is clear that maximal Hardy fields are Lioville closed. The authors have shown recently that they are ω-free, and there is an outline to try to prove that they are newtonian. The idea is solving "quasilinear" differential equations in the real of Hardy fields by constructing fixpoints. If that works it would settle Conjecture 1 from part II, with the consequence that all maximal Hardy fields are elementarily equivalent to $\mathbb{T}$.

Aso to Conjecture 2, there are promising partial results modulo Conjecture 1. For both the conjectures analytic arguments are needed in an essential way.

[accelero-summable transseries (some explanation)]

[$\mathbb{T}$ cannot be isomorphic to a maximal Hardy field, because $\mathbb{T}$ has countable cofinality]

[Liouville closed things cannot be spherically complete]