Notions of Domination Between Invariant Types

Rosario Mennuni

University of Leeds

Set Theoretic and Topological Methods in Model Theory
Tezpur, 7 September 2017

Overview

PhD project supervised by H.D. Macpherson and V. Mantova.

Everything is *very* work in progress. Comments, suggestions, corrections, etc. are extremely welcome.

Motivation

Notation

$\mathfrak{U} :=$ "monster" model of a complete first-order theory T with infinite models. I.e. $\mathfrak{U}$ realises all types over "small" sets, and types=orbits, i.e. $a \equiv_D b$ if and only if they are conjugate by the pointwise stabiliser $\mathrm{Aut}(\mathfrak{U}/D)$.

Motivation

Notation

$\mathfrak{U}$:= "monster" model of a complete first-order theory T with infinite models. I.e. $\mathfrak{U}$ realises all types over "small" sets, and types=orbits, i.e. $a \equiv_D b$ if and only if they are conjugate by the pointwise stabiliser $\mathrm{Aut}(\mathfrak{U}/D)$.

In [1] Haskell, Hrushovski and Macpherson associate to a monster model $\mathfrak{U}$ a certain quotient $\overline{\mathrm{Inv}}(\mathfrak{U})$ of global invariant types and prove this Ax-Kochen-Ershov-type result:

Theorem

In algebraically closed valued fields, $\overline{\mathrm{Inv}}(\mathfrak{U}) \cong \overline{\mathrm{Inv}}(k) \times \overline{\mathrm{Inv}}(\Gamma)$.

k := residue field, Γ := value group

Motivation

Notation

$\mathfrak{U}$:= "monster" model of a complete first-order theory T with infinite models. I.e. $\mathfrak{U}$ realises all types over "small" sets, and types=orbits, i.e. $a \equiv_D b$ if and only if they are conjugate by the pointwise stabiliser $\mathrm{Aut}(\mathfrak{U}/D)$.

In [1] Haskell, Hrushovski and Macpherson associate to a monster model $\mathfrak{U}$ a certain quotient $\overline{\mathrm{Inv}}(\mathfrak{U})$ of global invariant types and prove this Ax-Kochen-Ershov-type result:

Theorem

In algebraically closed valued fields, $\overline{\mathrm{Inv}}(\mathfrak{U}) \cong \overline{\mathrm{Inv}}(k) \times \overline{\mathrm{Inv}}(\Gamma)$.

k := residue field, Γ := value group

Goal

Study $\overline{\mathrm{Inv}}(\mathfrak{U})$ in general.

Invariant types

Definition

A type $p(x)$ is *A-invariant* iff for all $\varphi(x;y) \in L(\emptyset)$, whether $p(x) \vdash \varphi(x;b)$ or not depends only on $\mathrm{tp}(b/A)$.

Invariant types

Definition

A type $p(x)$ is *A-invariant* iff for all $\varphi(x;y) \in L(\emptyset)$, whether $p(x) \vdash \varphi(x;b)$ or not depends only on $\mathrm{tp}(b/A)$. A global type (i.e. in $S(\mathfrak{U})$) is *invariant* iff it is A-invariant for some *small* $A \subset \mathfrak{U}$. I.e. iff fixed (setwise) by the pointwise stabiliser $\mathrm{Aut}(\mathfrak{U}/A)$. "$A$ small" $= \mathfrak{U}$ is $|A|^+$-saturated and $|A|^+$-strongly homogeneous.

Invariant types

Definition

A type $p(x)$ is *A-invariant* iff for all $\varphi(x;y) \in L(\emptyset)$, whether $p(x) \vdash \varphi(x;b)$ or not depends only on $\mathrm{tp}(b/A)$. A global type (i.e. in $S(\mathfrak{U})$) is *invariant* iff it is A-invariant for some *small* $A \subset \mathfrak{U}$. I.e. iff fixed (setwise) by the pointwise stabiliser $\mathrm{Aut}(\mathfrak{U}/A)$. "$A$ small" = $\mathfrak{U}$ is $|A|^+$-saturated and $|A|^+$-strongly homogeneous.

Example

$T = \mathsf{DLO}$,

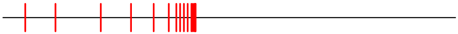
Invariant types

Definition

A type $p(x)$ is A-*invariant* iff for all $\varphi(x;y) \in L(\emptyset)$, whether $p(x) \vdash \varphi(x;b)$ or not depends only on $\mathrm{tp}(b/A)$. A global type (i.e. in $S(\mathfrak{U})$) is *invariant* iff it is A-invariant for some *small* $A \subset \mathfrak{U}$. I.e. iff fixed (setwise) by the pointwise stabiliser $\mathrm{Aut}(\mathfrak{U}/A)$. "$A$ small" $= \mathfrak{U}$ is $|A|^+$-saturated and $|A|^+$-strongly homogeneous.

Example

$T = \mathsf{DLO}$, A small,



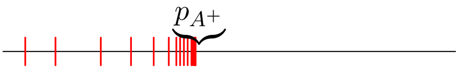
Invariant types

Definition

A type $p(x)$ is A-*invariant* iff for all $\varphi(x;y) \in L(\emptyset)$, whether $p(x) \vdash \varphi(x;b)$ or not depends only on $\mathrm{tp}(b/A)$. A global type (i.e. in $S(\mathfrak{U})$) is *invariant* iff it is A-invariant for some *small* $A \subset \mathfrak{U}$. I.e. iff fixed (setwise) by the pointwise stabiliser $\mathrm{Aut}(\mathfrak{U}/A)$. "$A$ small" $= \mathfrak{U}$ is $|A|^+$-saturated and $|A|^+$-strongly homogeneous.

Example

$T =$ DLO, A small, $p_{A^+} := \{x < b \mid b > A\} \cup \{x > c \mid c \not> A\}$.



Also, A-definable types are A-invariant.

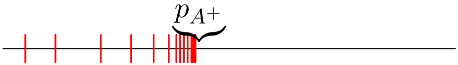
Invariant types

Definition

A type $p(x)$ is A-*invariant* iff for all $\varphi(x; y) \in L(\emptyset)$, whether $p(x) \vdash \varphi(x; b)$ or not depends only on $\mathrm{tp}(b/A)$. A global type (i.e. in $S(\mathfrak{U})$) is *invariant* iff it is A-invariant for some *small* $A \subset \mathfrak{U}$. I.e. iff fixed (setwise) by the pointwise stabiliser $\mathrm{Aut}(\mathfrak{U}/A)$. "$A$ small" = $\mathfrak{U}$ is $|A|^+$-saturated and $|A|^+$-strongly homogeneous.

Example

$T = \mathsf{DLO}$, A small, $p_{A^+} := \{x < b \mid b > A\} \cup \{x > c \mid c \not> A\}$.



Also, A-definable types are A-invariant.

Non-Example

$T = \mathsf{DLO}$, type of a cut with big cofinality on both sides.

Products

Fact

If $p \in S_x(\mathfrak{U})$ is invariant, for every $B \supseteq \mathfrak{U}$ it has a unique invariant extension $(p \mid B) \in S_x(B)$

Products

Fact

If $p \in S_x(\mathfrak{U})$ is invariant, for every $B \supseteq \mathfrak{U}$ it has a unique invariant extension $(p \mid B) \in S_x(B)$, namely:
$\varphi(x; b) \in p \mid B$ iff for $\tilde{b} \in \mathfrak{U}$ such that $b \equiv_A \tilde{b}$, we have $\varphi(x; \tilde{b}) \in p$.

Products

Fact

If $p \in S_x(\mathfrak{U})$ is invariant, for every $B \supseteq \mathfrak{U}$ it has a unique invariant extension $(p \mid B) \in S_x(B)$, namely:
$\varphi(x;b) \in p \mid B$ iff for $\tilde{b} \in \mathfrak{U}$ such that $b \equiv_A \tilde{b}$, we have $\varphi(x;\tilde{b}) \in p$.

Definition

$$\varphi(x,y;d) \in p(x) \otimes q(y) \overset{\text{def}}{\iff} \varphi(x;b,d) \in p \mid \mathfrak{U}b \qquad (b \vDash q)$$

Products

Fact

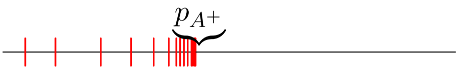
If $p \in S_x(\mathfrak{U})$ is invariant, for every $B \supseteq \mathfrak{U}$ it has a unique invariant extension $(p \mid B) \in S_x(B)$, namely:
$\varphi(x;b) \in p \mid B$ iff for $\tilde{b} \in \mathfrak{U}$ such that $b \equiv_A \tilde{b}$, we have $\varphi(x;\tilde{b}) \in p$.

Definition

$$\varphi(x,y;d) \in p(x) \otimes q(y) \overset{\text{def}}{\iff} \varphi(x;b,d) \in p \mid \mathfrak{U}b \qquad (b \vDash q)$$

Example

$$p_{A^+}(x) \otimes p_{A^+}(y) \vdash x < y \qquad (p_{A^+} := \{x < b \mid b > A\} \cup \{x > c \mid c \not> A\})$$



Products

Fact

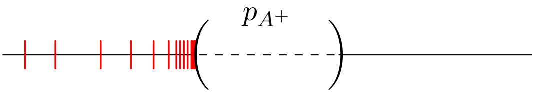
If $p \in S_x(\mathfrak{U})$ is invariant, for every $B \supseteq \mathfrak{U}$ it has a unique invariant extension $(p \mid B) \in S_x(B)$, namely:
$\varphi(x; b) \in p \mid B$ iff for $\tilde{b} \in \mathfrak{U}$ such that $b \equiv_A \tilde{b}$, we have $\varphi(x; \tilde{b}) \in p$.

Definition

$$\varphi(x, y; d) \in p(x) \otimes q(y) \overset{\text{def}}{\iff} \varphi(x; b, d) \in p \mid \mathfrak{U}b \qquad (b \vDash q)$$

Example

$p_{A^+}(x) \otimes p_{A^+}(y) \vdash x < y \qquad (p_{A^+} := \{x < b \mid b > A\} \cup \{x > c \mid c \not> A\})$



Products

Fact

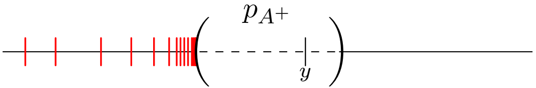
If $p \in S_x(\mathfrak{U})$ is invariant, for every $B \supseteq \mathfrak{U}$ it has a unique invariant extension $(p \mid B) \in S_x(B)$, namely:
$\varphi(x; b) \in p \mid B$ iff for $\tilde{b} \in \mathfrak{U}$ such that $b \equiv_A \tilde{b}$, we have $\varphi(x; \tilde{b}) \in p$.

Definition

$$\varphi(x, y; d) \in p(x) \otimes q(y) \overset{\text{def}}{\iff} \varphi(x; b, d) \in p \mid \mathfrak{U}b \qquad (b \vDash q)$$

Example

$$p_{A^+}(x) \otimes p_{A^+}(y) \vdash x < y \qquad (p_{A^+} := \{x < b \mid b > A\} \cup \{x > c \mid c \not> A\})$$



Products

Fact

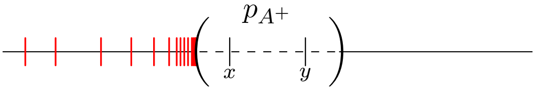
If $p \in S_x(\mathfrak{U})$ is invariant, for every $B \supseteq \mathfrak{U}$ it has a unique invariant extension $(p \mid B) \in S_x(B)$, namely:
$\varphi(x;b) \in p \mid B$ iff for $\tilde{b} \in \mathfrak{U}$ such that $b \equiv_A \tilde{b}$, we have $\varphi(x;\tilde{b}) \in p$.

Definition

$$\varphi(x,y;d) \in p(x) \otimes q(y) \overset{\text{def}}{\iff} \varphi(x;b,d) \in p \mid \mathfrak{U}b \qquad (b \models q)$$

Example

$$p_{A^+}(x) \otimes p_{A^+}(y) \vdash x < y \qquad (p_{A^+} := \{x < b \mid b > A\} \cup \{x > c \mid c \not> A\})$$



Products

Fact

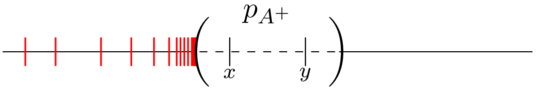
If $p \in S_x(\mathfrak{U})$ is invariant, for every $B \supseteq \mathfrak{U}$ it has a unique invariant extension $(p \mid B) \in S_x(B)$, namely:
$\varphi(x;b) \in p \mid B$ iff for $\tilde{b} \in \mathfrak{U}$ such that $b \equiv_A \tilde{b}$, we have $\varphi(x;\tilde{b}) \in p$.

Definition

$$\varphi(x,y;d) \in p(x) \otimes q(y) \overset{\text{def}}{\iff} \varphi(x;b,d) \in p \mid \mathfrak{U}b \qquad (b \vDash q)$$

Example

$p_{A^+}(x) \otimes p_{A^+}(y) \vdash x < y \qquad (p_{A^+} := \{x < b \mid b > A\} \cup \{x > c \mid c \not> A\})$



Fact

$\otimes$ is associative. It is commutative if T is stable (not in general).

HHM Domination

On global, invariant types (any finite number of variables), consider:

Definition

$p_x \sim q_y$ iff there are $M \subset \mathfrak{U}$ and $r \in S_{xy}(M)$ such that:

p, q are M-invariant, $r \supseteq (p \upharpoonright M) \cup (q \upharpoonright M)$,

HHM Domination

On global, invariant types (any finite number of variables), consider:

Definition

$p_x \sim q_y$ iff there are $M \subset \mathfrak{U}$ and $r \in S_{xy}(M)$ such that:

p, q are M-invariant, $r \supseteq (p \upharpoonright M) \cup (q \upharpoonright M)$,

$p \cup r \vdash q$ and $q \cup r \vdash p$

HHM Domination

On global, invariant types (any finite number of variables), consider:

Definition

$p_x \sim q_y$ iff there are $M \subset \mathfrak{U}$ and $r \in S_{xy}(M)$ such that:

p, q are M-invariant, $r \supseteq (p \restriction M) \cup (q \restriction M)$,

$p \cup r \vdash q$ and $q \cup r \vdash p$

Example (DLO)

$\mathrm{tp}(x > \mathfrak{U}) \sim \mathrm{tp}(y > z > \mathfrak{U})$

HHM Domination

On global, invariant types (any finite number of variables), consider:

Definition

$p_x \sim q_y$ iff there are $M \subset \mathfrak{U}$ and $r \in S_{xy}(M)$ such that:

p, q are M-invariant, $r \supseteq (p \restriction M) \cup (q \restriction M)$,

$p \cup r \vdash q$ and $q \cup r \vdash p$

Example (DLO)

$\mathrm{tp}(x > \mathfrak{U}) \sim \mathrm{tp}(y > z > \mathfrak{U})$ ("glue x and z",i.e. $r \coloneqq \{x = z\} \cup \dots$).

HHM Domination

On global, invariant types (any finite number of variables), consider:

Definition

$p_x \sim q_y$ iff there are $M \subset \mathfrak{U}$ and $r \in S_{xy}(M)$ such that:

p, q are M-invariant, $r \supseteq (p \restriction M) \cup (q \restriction M)$,

$p \cup r \vdash q$ and $q \cup r \vdash p$

Example (DLO)

$\mathrm{tp}(x > \mathfrak{U}) \sim \mathrm{tp}(y > z > \mathfrak{U})$ ("glue x and z",i.e. $r := \{x = z\} \cup \ldots$).

Example (ACF_0)

$p(x_1, \ldots, x_n) \sim q(y_1, \ldots, y_m) \iff \operatorname{tr deg}(x/\mathfrak{U}) = \operatorname{tr deg}(y/\mathfrak{U})$.

HHM Domination

On global, invariant types (any finite number of variables), consider:

Definition

$p_x \sim q_y$ iff there are $M \subset \mathfrak{U}$ and $r \in S_{xy}(M)$ such that:

p, q are M-invariant, $r \supseteq (p \restriction M) \cup (q \restriction M)$,

$p \cup r \vdash q$ and $q \cup r \vdash p$

Example (DLO)

$\operatorname{tp}(x > \mathfrak{U}) \sim \operatorname{tp}(y > z > \mathfrak{U})$ ("glue x and z",i.e. $r \coloneqq \{x = z\} \cup \ldots$).

Example (ACF_0)

$p(x_1, \ldots, x_n) \sim q(y_1, \ldots, y_m) \iff \operatorname{tr}\deg(x/\mathfrak{U}) = \operatorname{tr}\deg(y/\mathfrak{U})$.
(glue transcendence bases; recover the rest with one formula)

HHM Domination

On global, invariant types (any finite number of variables), consider:

Definition

$p_x \sim q_y$ iff there are $M \subset \mathfrak{U}$ and $r \in S_{xy}(M)$ such that:

p, q are M-invariant, $r \supseteq (p \restriction M) \cup (q \restriction M)$,

$p \cup r \vdash q$ and $q \cup r \vdash p$

Example (DLO)

$\mathrm{tp}(x > \mathfrak{U}) \sim \mathrm{tp}(y > z > \mathfrak{U})$ ("glue x and z",i.e. $r \coloneqq \{x = z\} \cup \ldots$).

Example (ACF_0)

$p(x_1, \ldots, x_n) \sim q(y_1, \ldots, y_m) \iff \operatorname{tr} \deg(x/\mathfrak{U}) = \operatorname{tr} \deg(y/\mathfrak{U})$.
(glue transcendence bases; recover the rest with one formula)

Definition

$\overline{\mathrm{Inv}}(\mathfrak{U})$ is the resulting quotient.

Is $\otimes$ a congruence?

Lemma

If $p_0 \sim p_1$, then $p_0 \otimes q \sim p_1 \otimes q$.

Is $\otimes$ a congruence?

Lemma

If $p_0 \sim p_1$, then $p_0 \otimes q \sim p_1 \otimes q$.

Corollary

If T is stable, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is well-defined.

Is $\otimes$ a congruence?

Lemma

If $p_0 \sim p_1$, then $p_0 \otimes q \sim p_1 \otimes q$.

Corollary

If T is stable, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is well-defined.

Open Question

Does $q_0 \sim q_1$ imply $p \otimes q_0 \sim p \otimes q_1$?

Is $\otimes$ a congruence?

Lemma

If $p_0 \sim p_1$, then $p_0 \otimes q \sim p_1 \otimes q$.

Corollary

If T is stable, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is well-defined.

Open Question

Does $q_0 \sim q_1$ imply $p \otimes q_0 \sim p \otimes q_1$?

In general, I still don't know. For some unstable theories, one can prove "by hand" that $\otimes$ is well-defined on $\overline{\mathrm{Inv}}(\mathfrak{U})$.

Is $\otimes$ a congruence?

Lemma

If $p_0 \sim p_1$, then $p_0 \otimes q \sim p_1 \otimes q$.

Corollary

If T is stable, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is well-defined.

Open Question

Does $q_0 \sim q_1$ imply $p \otimes q_0 \sim p \otimes q_1$?

In general, I still don't know. For some unstable theories, one can prove "by hand" that $\otimes$ is well-defined on $\overline{\mathrm{Inv}}(\mathfrak{U})$.

Fact

In $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$, if well-defined,

there is always a neutral element: the class of realised types

Is $\otimes$ a congruence?

Lemma

If $p_0 \sim p_1$, then $p_0 \otimes q \sim p_1 \otimes q$.

Corollary

If T is stable, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is well-defined.

Open Question

Does $q_0 \sim q_1$ imply $p \otimes q_0 \sim p \otimes q_1$?

In general, I still don't know. For some unstable theories, one can prove "by hand" that $\otimes$ is well-defined on $\overline{\mathrm{Inv}}(\mathfrak{U})$.

Fact

In $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$, if well-defined,

- there is always a neutral element: the class of realised types
- no other element has an inverse (a product of non-realised types is non-realised!).

Strongly Minimal Theories

Example

If T is strongly minimal, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes) \cong (\mathbb{N}, +)$.

(this probably generalises to countable $\aleph_1$-categorical theories; still not 100% sure)

Strongly Minimal Theories

Example

If T is strongly minimal, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes) \cong (\mathbb{N}, +)$.

(this probably generalises to countable $\aleph_1$-categorical theories; still not 100% sure)

The point is: the unique "generic" global type is not implied by a small subset of formulas. In this case, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is basically "counting the dimension".

Strongly Minimal Theories

Example

If T is strongly minimal, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes) \cong (\mathbb{N}, +)$.

(this probably generalises to countable $\aleph_1$-categorical theories; still not 100% sure)

The point is: the unique "generic" global type is not implied by a small subset of formulas. In this case, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is basically "counting the dimension".

Taking products corresponds to adding the dimension: by definition, if $(a, b) \models p \otimes q$, then $\dim(a/\mathfrak{U}b) = \dim(a/\mathfrak{U})$.

Dense Linear Orders

In DLO, classes different from $[\![0]\!]$ are given by a finite set of invariant cuts (i.e. small cofinality on exactly one side).

Dense Linear Orders

In DLO, classes different from $[\![0]\!]$ are given by a finite set of invariant cuts (i.e. small cofinality on exactly one side).
One can show "manually" that $\otimes$ is a congruence.
$(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is **commutative**: e.g. $p(x_0) \otimes p(y_0) \sim p(y_1) \otimes p(x_1)$ by gluing: $r \coloneqq \{x_0 = y_1 \land y_0 = x_1\} \cup \ldots$.

Dense Linear Orders

In DLO, classes different from $[\![0]\!]$ are given by a finite set of invariant cuts (i.e. small cofinality on exactly one side).
One can show "manually" that $\otimes$ is a congruence.
$(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is commutative: e.g. $p(x_0) \otimes p(y_0) \sim p(y_1) \otimes p(x_1)$ by gluing: $r := \{x_0 = y_1 \land y_0 = x_1\} \cup \dots$.
Also, every element is idempotent: e.g. if $p(x) = \mathrm{tp}(x > \mathfrak{U})$, then $p(x) \sim p(y) \otimes p(z)$ (seen before: glue x and z).

Dense Linear Orders

In DLO, classes different from $[\![0]\!]$ are given by a finite set of invariant cuts (i.e. small cofinality on exactly one side).
One can show "manually" that $\otimes$ is a congruence.
$(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is commutative: e.g. $p(x_0) \otimes p(y_0) \sim p(y_1) \otimes p(x_1)$ by gluing: $r := \{x_0 = y_1 \wedge y_0 = x_1\} \cup \dots$.
Also, every element is idempotent: e.g. if $p(x) = \mathrm{tp}(x > \mathfrak{U})$, then $p(x) \sim p(y) \otimes p(z)$ (seen before: glue x and z).
In fact, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is the free idempotent commutative monoid generated by the invariant cuts.

Dense Linear Orders

In DLO, classes different from $[\![0]\!]$ are given by a finite set of invariant cuts (i.e. small cofinality on exactly one side).
One can show "manually" that $\otimes$ is a congruence.
$(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is commutative: e.g. $p(x_0) \otimes p(y_0) \sim p(y_1) \otimes p(x_1)$ by gluing: $r := \{x_0 = y_1 \wedge y_0 = x_1\} \cup \ldots$.
Also, every element is idempotent: e.g. if $p(x) = \mathrm{tp}(x > \mathfrak{U})$, then $p(x) \sim p(y) \otimes p(z)$ (seen before: glue x and z).
In fact, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is the free idempotent commutative monoid generated by the invariant cuts.
The product corresponds to taking unions of sets of cuts.

Dense Linear Orders

In DLO, classes different from $[\![0]\!]$ are given by a finite set of invariant cuts (i.e. small cofinality on exactly one side).
One can show "manually" that $\otimes$ is a congruence.
$(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is commutative: e.g. $p(x_0) \otimes p(y_0) \sim p(y_1) \otimes p(x_1)$ by gluing: $r \coloneqq \{x_0 = y_1 \wedge y_0 = x_1\} \cup \dots$.
Also, every element is idempotent: e.g. if $p(x) = \mathrm{tp}(x > \mathfrak{U})$, then $p(x) \sim p(y) \otimes p(z)$ (seen before: glue x and z).
In fact, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is the free idempotent commutative monoid generated by the invariant cuts.
The product corresponds to taking unions of sets of cuts.
Note how this monoid has a compatible order structure: inclusion of subsets (will be relevant later).

Generic Equivalence Relation

Equivalence relation E with infinitely many infinite classes (and no finite classes).

Generic Equivalence Relation

Equivalence relation E with infinitely many infinite classes (and no finite classes). A set of generators for $\overline{\mathrm{Inv}}(\mathfrak{U})$ looks like this:

a single $\sim$-class $[\![0]\!]$ for realised types

Generic Equivalence Relation

Equivalence relation E with infinitely many infinite classes (and no finite classes). A set of generators for $\overline{\mathrm{Inv}}(\mathfrak{U})$ looks like this:

- a single $\sim$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E(x,a)\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\models E(a,b)$; corresponds to new points in an existing equivalence class

Generic Equivalence Relation

Equivalence relation E with infinitely many infinite classes (and no finite classes). A set of generators for $\overline{\mathrm{Inv}}(\mathfrak{U})$ looks like this:

- a single $\sim$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E(x,a)\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\vDash E(a,b)$; corresponds to new points in an existing equivalence class
- a single $\sim$-class $[\![p_g]\!]$, where $p_g := \{\neg E(x,a) \mid a \in \mathfrak{U}\}$; corresponds to new equivalence classes.

Generic Equivalence Relation

Equivalence relation E with infinitely many infinite classes (and no finite classes). A set of generators for $\overline{\mathrm{Inv}}(\mathfrak{U})$ looks like this:

- a single $\sim$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E(x,a)\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\vDash E(a,b)$; corresponds to new points in an existing equivalence class
- a single $\sim$-class $[\![p_g]\!]$, where $p_g := \{\neg E(x,a) \mid a \in \mathfrak{U}\}$; corresponds to new equivalence classes.

The product adds new points/new classes. So, if $\mathfrak{U}$ has κ equivalence classes,

$$\overline{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

More examples here

Commutativity?

A comment in [1] claimed that $\overline{\mathrm{Inv}}(\mathfrak{U})$ is always commutative. So was every example so far. In fact, it is false in general:

Example (M.)

$T :=$ the random graph. M small model. These types do not commute, even modulo $\sim$:

$$p := \{E(x,a) \mid a \in M\} \cup \{\neg E(x,b) \mid b \notin M\}$$
$$q := \{\neg E(x,a) \mid a \in M\} \cup \{E(x,b) \mid b \notin M\}$$

Proof Idea.

As $p \otimes q \vdash \neg E(x,y)$ and $q_z \otimes p_w \vdash E(z,w)$, gluing cannot work. But in the random graph there is not much more one can do. □

Commutativity?

A comment in [1] claimed that $\overline{\mathrm{Inv}}(\mathfrak{U})$ is always commutative. So was every example so far. In fact, it is false in general:

Example (M.)

$T :=$ the random graph. M small model. These types do not commute, even modulo $\sim$:

$$p := \{E(x,a) \mid a \in M\} \cup \{\neg E(x,b) \mid b \notin M\}$$

$$q := \{\neg E(x,a) \mid a \in M\} \cup \{E(x,b) \mid b \notin M\}$$

Proof Idea.

As $p \otimes q \vdash \neg E(x,y)$ and $q_z \otimes p_w \vdash E(z,w)$, gluing cannot work. But in the random graph there is not much more one can do. □

Example (Hrushovski)

In DLO plus a dense-codense predicate, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is not commutative. So, this is not a matter of NIP. More details here

Not The "Right" Notion?

One would hope $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ to have a compatible order, namely $[\![p]\!] \geq [\![q]\!]$ iff there are a small $M \subset \mathfrak{U}$ and $r \in S_{xy}(M)$ such that:

p, q are M-invariant, $r \supseteq (p \restriction M) \cup (q \restriction M)$,

$p \cup r \vdash q$, ~~$q \cup r \vdash p$~~

Not The "Right" Notion?

One would hope $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ to have a compatible order, namely $[\![p]\!] \geq [\![q]\!]$ iff there are a small $M \subset \mathfrak{U}$ and $r \in S_{xy}(M)$ such that:

p, q are M-invariant, $r \supseteq (p \upharpoonright M) \cup (q \upharpoonright M)$,

$p \cup r \vdash q$, ~~$q \cup r \vdash p$~~

Anyway, in DLO with a predicate, antisymmetry fails.

Maybe it is better to replace $p \sim q$ with $p \geq q \geq p$?

I.e. $p \cup r \vdash q$ and $q \cup s \vdash p$, for some possibly different $s \in S_{xy}(M)$.

Stable Case

In a stable theory, $\leq$ and $\sim$ can be expressed in terms of forking:

Definition (See e.g. [2])

$a \rhd_D b$ iff, for all c,

$$a \mathop{\smile\!\!\!\!\!\!\;|}_D c \Longrightarrow b \mathop{\smile\!\!\!\!\!\!\;|}_D c$$

Stable Case

In a stable theory, $\leq$ and $\sim$ can be expressed in terms of forking:

Definition (See e.g. [2])

$a \triangleright_D b$ iff, for all c,

$$a \downarrow_D c \Longrightarrow b \downarrow_D c$$

$p \triangleright_D q$ (p *dominates* q over D) iff there are $a \models p$ and $b \models q$ such that $a \triangleright_D b$

Stable Case

In a stable theory, $\leq$ and $\sim$ can be expressed in terms of forking:

Definition (See e.g. [2])

$a \rhd_D b$ iff, for all c,

$$a \downarrow_D c \Longrightarrow b \downarrow_D c$$

$p \rhd_D q$ (p *dominates* q over D) iff there are $a \models p$ and $b \models q$ such that $a \rhd_D b$

$p \bowtie_D q$ (p and q are *domination equivalent*) iff $p \rhd_D q \rhd_D p$

Stable Case

In a stable theory, $\leq$ and $\sim$ can be expressed in terms of forking:

Definition (See e.g. [2])

$a \triangleright_D b$ iff, for all c,

$$a \downarrow_D c \Longrightarrow b \downarrow_D c$$

$p \triangleright_D q$ (p *dominates* q over D) iff there are $a \models p$ and $b \models q$ such that $a \triangleright_D b$

$p \bowtie_D q$ (p and q are *domination equivalent*) iff $p \triangleright_D q \triangleright_D p$, i.e. there are $\underbrace{a}_{\models p} \triangleright_D \underbrace{b}_{\models q} \triangleright_D \underbrace{c}_{\models p}$

Stable Case

In a stable theory, $\leq$ and $\sim$ can be expressed in terms of forking:

Definition (See e.g. [2])

$a \rhd_D b$ iff, for all c,

$$a \downarrow_D c \Longrightarrow b \downarrow_D c$$

$p \rhd_D q$ (p *dominates* q over D) iff there are $a \models p$ and $b \models q$ such that $a \rhd_D b$

$p \bowtie_D q$ (p and q are *domination equivalent*) iff $p \rhd_D q \rhd_D p$, i.e. there are $\underbrace{a}_{\models p} \rhd_D \underbrace{b}_{\models q} \rhd_D \underbrace{c}_{\models p}$

$p \doteq_D q$ (p and q are *equidominant* over D) iff there are $a \models p$ and $b \models q$ such that $a \rhd_D b \rhd_D a$

Comparison with $\sim$

$\triangleright$ is well-behaved with non-forking extensions: drop ${}_D$ from $\triangleright_D$

Comparison with $\sim$

$\triangleright$ is well-behaved with non-forking extensions: drop $_D$ from $\triangleright_D$

Proposition (T stable)

$p \geq q \iff p \triangleright q$, and $p \sim q \iff p \doteq q$.

Comparison with $\sim$

$\triangleright$ is well-behaved with non-forking extensions: drop $_D$ from $\triangleright_D$

Proposition (T stable)

$p \geq q \iff p \triangleright q$, and $p \sim q \iff p \doteq q$. (so $p \geq q \geq p \iff p \bowtie q$)

Comparison with $\sim$

$\triangleright$ is well-behaved with non-forking extensions: drop $_D$ from $\triangleright_D$

Proposition (T stable)

$p \geq q \iff p \triangleright q$, and $p \sim q \iff p \doteq q$. (so $p \geq q \geq p \iff p \bowtie q$)

Remark

The proof uses **crucially** stationarity of types over models.

Comparison with $\sim$

$\triangleright$ is well-behaved with non-forking extensions: drop $_D$ from $\triangleright_D$

Proposition (T stable)

$p \geq q \iff p \triangleright q$, and $p \sim q \iff p \doteq q$. (so $p \geq q \geq p \iff p \bowtie q$)

Remark

The proof uses crucially stationarity of types over models.

Fact (See [3, Example 5.2.9])

Even in the stable case, $\bowtie$ and $\doteq$ are generally different.

Comparison with $\sim$

$\triangleright$ is well-behaved with non-forking extensions: drop $_D$ from $\triangleright_D$

Proposition (T stable)

$p \geq q \iff p \triangleright q$, and $p \sim q \iff p \doteq q$. (so $p \geq q \geq p \iff p \bowtie q$)

Remark

The proof uses crucially stationarity of types over models.

Fact (See [3, Example 5.2.9])

Even in the stable case, $\bowtie$ and $\doteq$ are generally different.

In almost all examples we saw before, $\bowtie$ coincides with $\sim$.

Comparison with $\sim$

$\triangleright$ is well-behaved with non-forking extensions: drop $_D$ from $\triangleright_D$

Proposition (T stable)

$p \geq q \iff p \triangleright q$, and $p \sim q \iff p \doteq q$. (so $p \geq q \geq p \iff p \bowtie q$)

Remark

The proof uses crucially stationarity of types over models.

Fact (See [3, Example 5.2.9])

Even in the stable case, $\bowtie$ and $\doteq$ are generally different.

In almost all examples we saw before, $\bowtie$ coincides with $\sim$. Exception: in DLO with a predicate, $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}, \otimes)$ is commutative. (in fact, it is the same as in DLO)

Comparison with $\sim$

$\triangleright$ is well-behaved with non-forking extensions: drop $_D$ from $\triangleright_D$

Proposition (T stable)

$p \geq q \iff p \triangleright q$, and $p \sim q \iff p \doteq q$. (so $p \geq q \geq p \iff p \bowtie q$)

Remark

The proof uses crucially stationarity of types over models.

Fact (See [3, Example 5.2.9])

Even in the stable case, $\bowtie$ and $\doteq$ are generally different.

In almost all examples we saw before, $\bowtie$ coincides with $\sim$.
Exception: in DLO with a predicate, $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}, \otimes)$ is commutative. (in fact, it is the same as in DLO)
The random graph **stays** non-commutative.

Question

Is $\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$ commutative under NIP?

Classical Results

In the superstable case, this is classical (e.g. [2]):

Theorem (T superstable)

$\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$ is a direct sum of copies of $\mathbb{N}$.

Classical Results

In the superstable case, this is classical (e.g. [2]):

Theorem (T superstable)

$\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$ is a direct sum of copies of $\mathbb{N}$.
Moreover, $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}, \otimes)$ is generated by $\{[\![p]\!] \mid p \text{ regular}\}$.

Classical Results

In the superstable case, this is classical (e.g. [2]):

Theorem (T superstable)

$\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$ is a direct sum of copies of $\mathbb{N}$.
Moreover, $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}, \otimes)$ is generated by $\{[\![p]\!] \mid p \text{ regular}\}$.

Superstability probably implies that $\sim$ (i.e. $\doteq$) and $\bowtie$ coincide.

Classical Results

In the superstable case, this is classical (e.g. [2]):

Theorem (T superstable)

$\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$ is a direct sum of copies of $\mathbb{N}$.
Moreover, $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}, \otimes)$ is generated by $\{[\![p]\!] \mid p \text{ regular}\}$.

Superstability probably implies that $\sim$ (i.e. $\doteq$) and $\bowtie$ coincide.

The behaviour of $\geq$ in general seems related to the existence of some kind of prime models (in the stable case, "prime a-models" are the way to go). This seems to hint that, maybe, o-minimal theories are a good context to investigate.

Classical Results

In the superstable case, this is classical (e.g. [2]):

Theorem (T superstable)

$\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$ is a direct sum of copies of $\mathbb{N}$.
Moreover, $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}, \otimes)$ is generated by $\{[\![p]\!] \mid p \text{ regular}\}$.

Superstability probably implies that $\sim$ (i.e. $\doteq$) and $\bowtie$ coincide.

The behaviour of $\geq$ in general seems related to the existence of some kind of prime models (in the stable case, “prime a-models” are the way to go). This seems to hint that, maybe, o-minimal theories are a good context to investigate.
Also, some kind of Omitting Types Theorem would surely help.

Other Generalisations

We said that, if T is stable, we have an equivalence between

domination in the sense of forking: $\triangleright$

"implication modulo a small type": $\geq$

Other Generalisations

We said that, if T is stable, we have an equivalence between

domination in the sense of forking: $\triangleright$

"implication modulo a small type": $\geq$

While we have been talking about generalising $\geq$, there has been some work in generalising $\triangleright$:

(super)simple context (see e.g. [3]); problem: lack of stationarity ($\doteq$ is not even transitive, if considering types over arbitrary sets; maybe everything is fine when considering global types)

Other Generalisations

We said that, if T is stable, we have an equivalence between

domination in the sense of forking: $\triangleright$

"implication modulo a small type": $\geq$

While we have been talking about generalising $\geq$, there has been some work in generalising $\triangleright$:

(super)simple context (see e.g. [3]); problem: lack of stationarity ($\doteq$ is not even transitive, if considering types over arbitrary sets; maybe everything is fine when considering global types)

Onshuus and Usvyatsov: $\triangleright$ with þ-forking in (super)rosy theories (e.g. [4, 5]).

Other Generalisations

We said that, if T is stable, we have an equivalence between

- domination in the sense of forking: $\triangleright$
- "implication modulo a small type": $\geq$

While we have been talking about generalising $\geq$, there has been some work in generalising $\triangleright$:

- (super)simple context (see e.g. [3]); problem: lack of stationarity ($\doteq$ is not even transitive, if considering types over arbitrary sets; maybe everything is fine when considering global types)
- Onshuus and Usvyatsov: $\triangleright$ with þ-forking in (super)rosy theories (e.g. [4, 5]).

Another notion on the $\leq$ side is $\leq_{\mathrm{RK}}$ More on this here

Questions

Some of the main questions I would like to answer are:

Is $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}), \otimes)$ always well defined? If not, when is this the case?

When does $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}, \otimes)$ depend on $\mathfrak{U}$? Any unstable examples where it doesn't?

When is it commutative? Is this always the case in NIP theories?

Does bi-interpretation result in isomorphism of $\overline{\mathrm{Inv}}(\mathfrak{U})$? (Known: mutual interpretation is not enough)

The size of r is not in general bounded (DLO). Can one at least bound the number of formulas involving both x and y?

What can one say on $\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$ based on properties of T?

Conversely, what properties of T can be recovered by looking at $\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$?

What properties are preserved by $\bowtie$? E.g. regularity is not, finite satisfiability probably is.

Does anything like this make sense for measures?

Is there any sensible topology to put on $\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$?

Questions

Some of the main questions I would like to answer are:

Is $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}), \otimes)$ always well defined? If not, when is this the case?

When does $(\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}, \otimes)$ depend on $\mathfrak{U}$? Any unstable examples where it doesn't?

When is it commutative? Is this always the case in NIP theories?

Does bi-interpretation result in isomorphism of $\overline{\mathrm{Inv}}(\mathfrak{U})$? (Known: mutual interpretation is not enough)

The size of r is not in general bounded (DLO). Can one at least bound the number of formulas involving both x and y?

What can one say on $\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$ based on properties of T?

Conversely, what properties of T can be recovered by looking at $\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$?

What properties are preserved by $\bowtie$? E.g. regularity is not, finite satisfiability probably is.

Does anything like this make sense for measures?

Is there any sensible topology to put on $\overline{\mathrm{Inv}}(\mathfrak{U})_{\bowtie}$?

Thanks for listening!

Bibliography

(this is not a proper bibliography, it's just a list of the sources mentioned in the previous slides)

D. Haskell, E. Hrushovski and D. Macpherson,
Stable Domination and Independence in Algebraically Closed Valued Fields,
Lecture Notes in Logic 30, Cambridge University Press 2007

B. Poizat,
A Course in Model Theory,
Universitext, Springer 2000

F.O. Wagner,
Simple Theories,
Mathematics and Its Applications 503, Kluwer Academic Publishers 2000

A. Onshuus, A. Usvyatsov,
Stable domination and weight,
Annals of Pure and Applied Logic 162 (2011), 544-560

A. Onshuus, A. Usvyatsov,
Thorn Orthogonality and Domination in Unstable Theories,
Fundamenta Mathematicae 214 (2011), 241–268

More examples: Branches

Example

Let T be the theory in the language $\{P_\sigma \mid \sigma \in 2^{<\omega}\}$ asserting that every point belongs to every $P_{\eta\restriction n}$ for exactly one $\eta \in 2^\omega$. Then $\overline{\mathrm{Inv}}(\mathfrak{U}) \cong \bigoplus_{2^{\aleph_0}} \mathbb{N}$.

Basically, $\overline{\mathrm{Inv}}(\mathfrak{U})$ here is counting how many new points are in a "branch".

More Examples: Cross-cutting Equivalence Relations

$T_n :=$ n generic equivalence relations E_i; intersection of classes of different E_i always infinite. Here $(\overline{\text{Inv}}(\mathfrak{U}), \otimes)$ is generated by:

- a single $\sim$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E_i(x, a) \mid i < n\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\vDash \bigwedge_{i<n} E_i(a, b)$; corresponds to new points in E_i-relation with a for all i
- For each $i < n$, a class $[\![p_i]\!]$ saying x is in a new E_i class, but in existing E_j-classes for $j \neq i$ (does not matter which)

So

$$\overline{\text{Inv}}(\mathfrak{U}) \cong \prod_{i<n} \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

Why $\prod$ instead of $\bigoplus$? If we allow, say, $\aleph_0$ equivalence relations, then

$$\overline{\text{Inv}}(\mathfrak{U}) \cong \prod_{i<\aleph_0} \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

Hrushovski's Counterexample

Example (Hrushovski)

In DLO plus a dense-codense predicate P, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is not commutative.

Proof idea.

Let $p(x) := \{P(x)\} \cup \{x > \mathfrak{U}\}$ and $q(y) := \{\neg P(x)\} \cup \{y > \mathfrak{U}\}$.
Then p, q do not commute, even modulo $\sim$.
The predicate P forbids to "glue" variables. One will be "left behind": e.g. if $r \vdash x_0 < y_0 < y_1 < x_1$, knowing that $y_1 > \mathfrak{U}$ does not imply $x_0 > \mathfrak{U}$. □

In this case, for each cut C there are generators $[\![p_{C,P}]\!]$ and $[\![p_{C,\neg P}]\!]$, with relations

$$[\![p_{C,P}]\!] \otimes [\![p_{C,P}]\!] = [\![p_{C,\neg P}]\!] \otimes [\![p_{C,P}]\!] = [\![p_{C,P}]\!]$$

(same relations swapping P and $\neg P$)

$$[\![p_{C_0,-}]\!] \otimes [\![p_{C_1,-}]\!] = [\![p_{C_0,-}]\!] \otimes [\![p_{C_1,-}]\!] \text{ whenever } C_0 \neq C_1.$$

RK-Domination

Another notion classically studied is (see e.g. [2]):

Definition

$p \geq_{\mathrm{RK}} q$ iff every model realising p realises q.

This behaves best in totally transcendental theories (because of prime models). It corresponds $p(x) \cup \varphi(x, y) \vdash q(y)$.
But even there, it is *not* true that every type decomposes as a product of $\geq_{\mathrm{RK}}$-minimal types.
A classical example where $\triangleright$ differs from $\geq_{\mathrm{RK}}$: generic equivalence relation with a bijection s such that $\forall x\ E(x, s(x))$.

Back