Unofficial Notes

Taken during the Graduate Student and Post Doc Mini-Courses of the Thematic Program on Model Theory at the Center for Mathematics at Notre Dame.

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Disclaimer

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Info

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Rosario Mennuni

[1] And it is not clear if they will ever be.

Chapter 1

Continuous Model Theory (Bradd Hart)

1.1 Ultraproducts and Metric Structures (13/06 - 09:30)

Slides and references are available on the summer school site. The goal of this course is to give a rough overview of the basics of Continuous Model Theory. We suppose knowledge of classical model theory at a graduate level. There have been a number of tutorial on continuous model there and there should be slides and full courses online (for instance there was something in Buenos Aires).

The concept to consider a logic with continuous truth values goes back a long way (Tarski, 20's). There is a book by Chang and Keisler called "Continuous Model Theory". Henson developed *positive bounded logic* to study Banach spaces. This more or less starts the subject (Chang and Keisler were probably too general). Then *CATS* were invented (another thing).

To get problems to work on it seems fruitful to see where ultraproducts are used in mathematics. For instance in Banach spaces and operator algebras (C*-algebras), and in geometry (asymptotic cones). That's why we're starting with ultraproducts.

Definition 1.1 (Filter)**.**

Lemma 1.2. $G \subseteq \mathscr{P}(X)$ is contained in a filter iff it has the FIP.

Definition 1.3 (Ultrafilter)**.**

Lemma 1.4. Ultrafilters are maximal filters and every filter can be extended to an ultrafilter.

Definition 1.5 (Ultralimits)**.**

Lemma 1.6. If $\bar{r}\colon I \to \mathbb{R}$ is bounded then

- $\lim_{i\to\mathcal{U}} r_i$ exists and is unique
- $\lim_{i\to\mathcal{U}} r_i = \inf\{B \mid \{i \in I \mid r_i < B\} \in \mathcal{U}\}$
- $\lim_{i\to\mathcal{U}} r_i = \sup\{B \mid \{i \in I \mid r_i > B\} \in \mathcal{U}\}$

Fix I, $\mathcal{U} \in \beta I$ and $(X_i, d_i)_{i\in I}$ a uniformly bounded collection of metric spaces, i.e. all with diameter $\leq B$.

Lemma 1.7. *$d(x, y) = \lim_{i\to\mathcal{U}} d_i(x_i, y_i)$ is a pseudo-metric on $\prod_{i\in I} X_i$*

Definition 1.8. The ultraproduct of the X_i wrt $\mathcal{U}$ is the product quotienting by d. We denote the ultrapower of X as $X^{\mathcal{U}}$.

Remark 1.9. Uniform boundedness is not entirely necessary, and is dropped in some applications (e.g. asymptotic cones). In those cases you work with pointed spaces.

Exercise 1.10. For a compact metric space, $X^{\mathcal{U}} \cong X$ as metric spaces. (this is useful with $X = [0, 1]$ so you don't change "truth values" when taking ultraproducs).

Exercise 1.11. If X_i is an I-indexed family of uniformly bounded complete metric spaces then $\prod_{\mathcal{U}} X_i$ is complete.

Exercise 1.12. Show that for any family of uniformly bounded metric spaces $(X_n \mid n \in \mathbb{N})$, if $\mathcal{U}$ is nonprincipal then $\prod_{\mathcal{U}} X_n$ is complete. (this is an instance of saturation)

Exercise 1.13. Show that this definition of ultraproduct is the same as the classical one with a 2-valued metric.

Remark 1.14. If we want to add structure, say with a single additional function f, we want our notions to be preserved under ultraproducts. How do we define f on the ultraproduct? We must require that f is uniformly continuous, in order to have a unique extension starting from a dense set, the image of the diagonal map.

Proof. Suppose f is not continuous at $x \in X$ and fix $\varepsilon > 0$ such that for all n there is y_n such that $d(x, y_n) < 1/n$ and $d(f(x), f(y_n)) \geq \varepsilon$. Then, in the ultraproduct $X^{\mathcal{U}}$ we have $d(x, y) = 0$, and $d(f(x), f(y)) \neq 0$, so f is not well-defined on the quotient. This shows the need for continuity.

For uniform continuity, repeat the argument with $d(x_n, y_n) < 1/n$ and $d(f(x_n), f(y_n)) \geq \varepsilon$. □

For the same reason, if we have a relation $R\colon X \to Y$ its range Y must be compact and R must be uniformly continuous. We wll assume WLOG that we always have $Y = [0, 1]$ (this is because of some embedding theorem inside the *Hilbert cube.*).

Definition 1.15. A language $\mathcal{L}$ consists of

- A set $\mathcal{S}$ of *sorts*
- A family $\mathcal{F}$ of function symbols $f \in \mathcal{F}$ equipped with domain $\prod_{i \leq n} S_i$ and range S, where S_i, S are sorts. Moreover we specify a continuity modulus $\delta_i^f : [0,1] \to [0,1]$ (which is not though as a function symbols but as a "real" function.
- A family $\mathcal{R}$ of relation symbols $R \in \mathcal{R}$ equipped with a domain $\prod_{i \leq n} S_i$ and a range K_R, where the S_i are sorts and R_K is a closed interval.
- For each $S \in \mathcal{S}$, one special relation symbol $d_S \colon S \times S \to [0, B_s]$. Its continuity moduli are the identity functions.

Definition 1.16. A metric structure $\mathcal{M}$ interprets a language with [did not write all the details]

- An $\mathcal{S}$-indexed family of complete bounded metric spaces $(S^{\mathcal{M}}, d_S^{\mathcal{M}})$ with bound B_S for $S \in \mathcal{S}$.
- A family of functions $(f^{\mathcal{M}} \mid f \in \mathcal{F})$ with the right domain, range and modulus of continuity.
- Same for relations.

Dropping completeness does not change the logic (we will see). Some text do and call that pre-structure, then complete it and call it a structure.

Example 1.17. Complete bounded metric spaces (X, d) with no additional structure.

Example 1.18. Ordinary first order-structures with the discrete metric. Functions are automatically uniformly continuous and Relations become 2-valued functions.

Remark 1.19. In continuous logic the value for "true" is 0, not 1. That's arbitrary, but notice that $x = y$ iff $d(x, y) = 0$. Or because you want to look at zero-sets.

Hilbert spaces (we're interested in complex ones) can be seen as metric structures. Let

- B_n be the ball $B(0, n)$. It will be a sort.
- Take the inclusion maps $B_n \to B_m$ for $n \leq m$.
- 0 a constant in B_1
- For every $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, a unary function $\lambda_n \colon B_n \to B_m$ for suitable m (scalar multiplication)

- Same issue of "changing balls" for the sum.
- The inner product is complex valued, so it is done with two predicats, one for the real part and one for the imaginary part (and you need to define it on each $B_n \times B_m$).

It is easy to check that all this stuff is uniformly continuous. Notice that the realtions have bounded range since they are restricted to bounded balls.

Question 1.20. Does every such a structure arise from an Hilbert space H?

Of course not. Some work is required to have this to work out well, for instance to have the two correspondences between Hilbert spaces and (suitable class of metric structures) inverses of each other.

Definition 1.21. The ultraproduct of metric structures are defined taking the ultraproduct of the sorts and the ultralimits of functions and of relation symbols.

Exercise 1.22 (Important)**.** Let $(X_i, d_i)_{i\in I}$ is uniformly bounded, $\mathcal{U} \in \beta I$ and f_i is an n-ary uniformly continuous relation with a fixed continuity modulus and range in K, a compact interval. Prove that if (Y, d, f) is the ultraproduct then

$$\sup_{x\in Y} f(x, a_2, \ldots, a_n) = \lim_{i\to\mathcal{U}} \sup_{x} f((a_2)_i, \ldots, (a_n)_i)$$

1.2 The Basics (13/06 - 14:00)

Terms are defined inductively as in classical logic, apart from the fact that you have to keep track of moduli of continuity.

Definition 1.23.

If R is a relation symbol and $\tau_1, \ldots, \tau_n$ are terms[1] then $R(\tau_1, \ldots \tau_n)$ is a formula.

If $\varphi_i(x)$ is a formula with range K_{φ_i} and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function then $f(\varphi_1, \ldots, \varphi_n)$ is a formula with range $f(\prod_{i\leq n} K_{\varphi_i}$

If φ is a formula and x is a variable[2] then $\sup_x \varphi$ and $\inf_x \varphi$ are both formulas The sort of x is removed from the domain and continuity moduli for remaining variables stay the same. These play the role of quantifiers.

Definition 1.24. The interpretation in a structure is defined as follows:

- Terms are interpreted as usual

[1] Remember to track continuity moduli!
[2] Variables have the identity as continuity module.

- $R(\tau_1(x), \ldots, \tau_n(x))$ is interpreted as $R^{\mathcal{M}}(\tau_1^{\mathcal{M}}(a), \ldots, \tau_n^{\mathcal{M}}(a))$
- Continuous functions are handled as you expect
- So do sup and inf.

Remark 1.25. Up to taking $|\cdot|$ we can assume $\varphi(x) \geq 0$. What does it mean that $\sup_x(\varphi(x)) = 0$? Well, that it is 0 identically, so sup plays the role that $\forall$. The analogy goes much further, in the sense that for sup-axiomatized theories they work something like $\forall$-axiomatized theories, etc. For $\inf_x \varphi(x) = 0$ the situation is *not* the same as with $\exists$, because this only means that you have, for all $n \in \mathbb{N}$, some x_n such that $\varphi(x_n) < 1/n$.

Proposition 1.26. The interpretation of terms in $\mathcal{M}$ are uniformly continuous functions, and the modulus of continuity only depends on the language, not on the structure.

Proposition 1.27. The interpretation of formulas are uniformly continuous functions with domains, range, and continuous modulus specified by the definition.

Definition 1.28. A sentence is a formula with no free variables.

As a consequence of the above propositions, a sentence takes value in a compact interval depending on $\mathcal{L}$, and not on $\mathcal{M}$.

Remark 1.29. As for the pure model theory, we could always take the interval to be $[0, 1]$. Anyway, for application it is more comfortable to be able to choose the best suited interval.

Definition 1.30. Le $\mathrm{Sent}_{\mathcal{L}}$ be the set of sentences of $\mathcal{L}$. The theory of a structure $\mathcal{M}$ is the function $\mathrm{Th}(\mathcal{M})\colon \mathrm{Sent}_{\mathcal{L}} \to \mathbb{R}$ defined by the interpretation $\mathrm{Th}(M)(\varphi) = \varphi^{\mathcal{M}}$.

Remark 1.31. Notice that $\mathrm{Sent}_{\mathcal{L}}$ is a vector space and $\mathrm{Th}(\mathcal{M})$ is a linear functional which is determined by its kernel. Therefore we also say that $\{\varphi \in \mathrm{Sent}_{\mathcal{L}} \mid \varphi^{\mathcal{M}} = 0\}$ is the theory of $\mathcal{M}$.

Definition 1.32. An $\mathcal{L}$-theory T is something contained[3] in $\mathrm{Th}(\mathcal{M})$ for some $\mathcal{M}$.

So we will also drop the "$= 0$" and, for instance, say that $\sup_x \varphi(x)$ is a sentence.

Let's axiomatize Hilbert spaces. Let us try and write down some formulas that evaluate to 0 in all Hilbert spaces.

[3] In the sense of the previous remark.

- There are universal sentences expressing that we have a complex inner product space, for instance

$$\sup_{x\in B_1}\sup_{y\in B_1} d_{B_2}(x +_{1,1} y, y +_{1,1} x)$$

- There is the relationship between the inner product and the metric

$$\sup\sup(d(x,y)^2) - \mathrm{re}(\langle x-y, x-y\rangle)$$

- Also[4] $\sup(d(x,0) \dot{-} 1)$.
- Consider the sentence, for all $n \in \mathbb{N}$,

$$\sup_{x\in B_n} \min\{1 \dot{-} d(x,0), \inf_{y\in B_1} d(x, i(y))\}$$

  This says that something of norm ≤ 1 in B_n comes from an element of B_1.[5]

- There are sentences expressing that an Hilbert space is infinite-dimensional (exercise)

Proposition 1.33. There is only one separable model of these axioms.

Proposition 1.34. This theory is complete.

Theorem 1.35 (Łoś Theorem for Continuous Logic)**.** In ultraproducts $\varphi^{\mathcal{M}}(a) = \lim_{i\to\mathcal{U}} \varphi^{\mathcal{M}_1}(a_i)$

Remark 1.36. Notice that, in contrast with the discrete world, we can have a sentence that holds in the ultraproduct but it only "almost holds" in most structures: it only has to be, for each n, $\mathcal{U}$-almost-everywhere $\leq 1/n$.

Definition 1.37. A set of sentences is satisfied in a strucutre if every one of them is satisfied. Finite satisfiability is analogously defined. For Σ a set of sentences and $\varepsilon > 0$ we defin the ε-approximation of Σ as $\{|\varphi| \dot{-} \varepsilon \mid \varphi \in \Sigma\}$ MISSING

Theorem 1.38 (Compactness Theorem for Continuous Logic)**.** TFAE

- Σ is satisfiable
- Σ is finitely satisfiable
- Σ is approximately finitely satisfiable.

[4] $x \dot{-} y$ is $\max 0, x-y$.

[5] Otherwise there are problems: take orthogonal things in infinite dimension, and take 1/2 of one of these guys, when you turn this into an Hilbert space and then come back...

We have a lot of formulas, due to how many connectives (continuous functions) we have. This is not how we should think about them. We should think about them in topological terms. So define

Definition 1.39. The distance between $\varphi(x)$ and $\psi(x)$ is[6]

$$\sup\{|\varphi^{\mathcal{M}}(a) - \psi^{\mathcal{M}}(a)| \mid \mathcal{M} \text{ an } \mathcal{L}\text{-formula}, a \in \mathcal{M}\}$$

We will call this space $\mathcal{F}_{\bar{\mathcal{S}}}$. This can also be relatived to all structures satisfying a fixed theory.

Proposition 1.40. The previous is a pseudo-metric.

We're interested in the *density character* of the set of formulas.

Definition 1.41. The density character of a topological space X is the infimum of the cardinality of a dense subset of X, written $\chi(X)$.

An infinite separable space has countable density character.

Proposition 1.42. If $\mathcal{L}$ is countable, then for any tuple of sorts $\bar{\mathcal{S}}$ we have that $\mathcal{F}_{\bar{\mathcal{S}}}$ is separable.

Definition 1.43. Substructures are defined as expected (remember to take them closed). Submodels are elementary if we always have $\varphi^{\mathcal{M}}(a) = \varphi^{\mathcal{N}}(a)$. Embedding and elementary embeddings are defined as usual. $\mathrm{Mod}(T)$ is the category of models of T with elementary maps as morphisms. We call such a class an elementary class.

Remark 1.44. By Łoś, the diagonal embedding $d\colon \mathcal{M} \to \mathcal{M}^{\mathcal{U}}$ is elementary.

Proposition 1.45 (Tarski-Vaught's test)**.** If $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M}$ is an elementary submodel if for every $\varphi(x, y)$, $r \in \mathbb{R}$ and $a \in \mathcal{M}$, if $\inf_x(\varphi(x, a))^{\mathcal{N}} < r$ then there is $b \in \mathcal{M}$ such that $(\varphi(b, a))^{\mathcal{N}} < r$.

Theorem 1.46 (Downward Löwenheim-Skolem)**.** If $\mathcal{N}$ is an $\mathcal{L}$-structure and A is a subset of $\mathcal{N}$ then there is an elementary $\mathcal{M} \subseteq \mathcal{N}$ that contains A and for every sort S we have $\chi(S^{\mathcal{M}}) \leq \chi(\mathcal{L}) + \chi(A)$.

Theorem 1.47. If $\mathcal{C}$ is a class of $\mathcal{L}$ structures, TFAE

1. $\mathcal{C}$ is an elementary class
2. $\mathcal{C}$ is closed under isomorphisms, ultraproducts and elementary submodels.
3. $\mathcal{C}$ is closed under isomorphisms, ultraproducts and ultraroots.

[6] Actually the sup will be attained by an ultraproduct: remember that truth values are in $\mathbb{R}$.

Theorem 1.48. Continuous first order logic is the maximal logic on metric structures which satisfies compactness, the downward Löwenheim-Skolem and unions of elementary chains.

Remark 1.49. This theorem with classical logic does not mention the last clause. Why? The point is that some contructions in continuous logic are more delicate. The problem is connecting different sorts: stuff needs to be uniformly continuous.

"We're about to do real model theory, and everybody knows that real model theory starts with types, so tomorrow we'll start with types".

1.3 Discussion

I'm not writing everything (also because some questions were unhearable due to volume issues).

Question 1.50. Can you tell us about asymptotic cones?

Take the Cayley graph of a group with the metric given setting $d(x, y)$ equal to the minimal length of $x^{-1}y$. Consider (G, d, e), where e is the identity. The asymptotic cone is obtained by taking $X_n = (G, d/n, e)$ and taking a non-principal ultraproduct (think, for instance, of $X_n = B_n$). Notice that every element in the ultraproduct is at bounded distance from the origin, because we have the $(x_n \mid n < \omega)$ such that $\lim_{\mathcal{U}} \frac{d}{n}(x_n, e) < \infty$.

For the free group F_2, no matter who is $\mathcal{U}$, you always get the same metric space (should be called R-tree or something). In order for this to happen "something should be stable" (this can be made precise, at least on a conjecture level). (how many ultraproducts there are is very related to stability)

Question 1.51. Is there a notion of higher order continuous logic?

"Weakly yes". This is related to duals. There is a recent paper where they do something like this in special cases.

Question 1.52. So a theory is a functional. How does the space of such functionals look like?

They are continuous, and they are very few (in the sense that they have to respect some closure properties: thik of connectives...). It is compact with the logic (i.e. weak*) topology.

Question 1.53. What about embedding continuous logic into classical logic?

You can do what you think you should do: you add predicates for everything, etc. If you tae an ω-saturated and look at the hyperimaginaries (quotient by type-definable equivalence relations) you have metrics in hyperimaginaries living in this discrete structure. Every metric structures can be thought like that.

1.4 Definable Sets (14/06 - 10:30)

Fix T and $\mathcal{L}$. Consider partial functions p on the space of formulas $\mathcal{F}_{\bar{\mathcal{S}}}$ to $\mathbb{R}$.

Definition 1.54. p is a partial type if there is a model $\mathcal{M}$ of T and $a \in \mathcal{M}$ of the appropriate sort such that $p(\varphi) = \varphi^{\mathcal{M}}(a)$ for all $\varphi \in \mathrm{dom}(p)$. A type is complete if its domain is $\mathcal{F}_{\bar{\mathcal{S}}}$.

Proposition 1.55. p is a type iff it is finitely satisfied, i.e. if the restriction to every finite subset of its domain is a type.

Remark 1.56. A complete type is a linear functional on $\mathcal{F}_{\bar{\mathcal{S}}}$.

let $S_{\bar{\mathcal{C}}}(T)$ be the space of complete types of sort $\bar{\mathcal{S}}$. The logic topology on $S_{\bar{\mathcal{S}}}(T)$ is the restriction of the weak* topology on the dual of $\mathcal{F}_{\bar{\mathcal{S}}}$. Equivalently, the basic open sets are the ones of the form

$$\{p \in S_{\bar{\mathcal{S}}}(T) \mid p(\varphi) < r\}$$

Proposition 1.57. The logic topology is compact Hausdorff and the functions $p \mapsto p(\varphi)$ are continuous.

Proposition 1.58. Tfae:

1. f is a continuous function $S_{\bar{\mathcal{S}}}(T) \to \mathbb{R}$
2. f is the uniform limit of functions of the form f_φ, i.e. for every n there is φ_n such that for all p we have $|f(p) - p(\varphi_n)| \leq 1/n$.

Definition 1.59. A Cauchy sequence of formulas $\bar{\varphi}$ will be called a *definable predicate* and interpreted in an $\mathcal{L}$-structure $\mathcal{M}$ by

$$\bar{\varphi}^{\mathcal{M}}(a) = \lim_{n\to\infty} \varphi_n^{\mathcal{M}}(a).$$

In other words we are extending the notion of formula to the Banach space generated by $\mathcal{F}_{\bar{\mathcal{S}}}$.

Example 1.60. $\sum \frac{\varphi_n}{B_n 2^n}$ is a definable predicate.

Fix T. We define a metric on $S_{\bar{\mathcal{S}}}(T)$ this way: $d(p,q)$ is the infimum of $d^{\mathcal{M}}(a,b)$ where $\mathcal{M}$ ranges over all models of T and a, b are realizations respectively of p and q. Then d is computed as the maximum of the d_S as S ranges over the sorts in $\bar{\mathcal{S}}$.

Proposition 1.61. This defines a metric and that by compactness $d(p,q)$ is always realized.

Proposition 1.62. The metric topology refines the logic topology.

Question 1.63. When do they coincide?

Question 1.64. When do they coincide at a point[7]?

Lemma 1.65. If $F, G\colon X \to [0,1]$ are functions such that

$$\forall \varepsilon > 0 \exists \delta > 0\, \forall x \in X (F(x) \leq \delta \Rightarrow G(x) \leq \varepsilon)$$

then there is an increasing continuous $\alpha\colon [0,1] \to [0,1]$ such that $\alpha(0) = 0$ and $\forall x \in X\ g(x) \leq \alpha(F(x))$.

Notice that α is a connective. In other words if we see an approximation result like the one above between F and G we can turn it into a formula in continuous logic.

Definition 1.66. For a definable predicate $\varphi(x)$, the zero set of φ in $\mathcal{M}$ is

$$\{a \in \mathcal{M} \mid \varphi^{\mathcal{M}}(a) = 0\}$$

If X is a non-empty closed subset of some product of sorts we call $P(x) = d(x, X) = \inf\{d(x,y) \mid y \in X\}$ the distance predicate for X.

In classical logic there is no problem in relativizing statements to a set (bounded quantification), as in $\forall x\ \varphi(x) \to \psi(x)$. How do you do this in continuous logic? It cannot always be done. So, how do the "good" subsets look?

Definition 1.67. Suppose we have a theory T in $\mathcal{L}$ and S_i, for $i \leq n$, are sorts in $\mathcal{L}$. We call an assignment to every model $\mathcal{M}$ of T, a closed subset $X^{\mathcal{M}}$ of $\prod_{j\leq n} S_j^{\mathcal{M}}$ a uniform assignment realtive to the theory T. This assignment $\mathcal{M} \mapsto X^{\mathcal{M}}$ is called a *definable set* if, for all formulas $\psi(x,y)$, the following functions, defined for all $\mathcal{M}$ models of T, are definable predicates:

$$\sup_{x\in X^{\mathcal{M}}} \psi^{\mathcal{M}}(x,y) \qquad\qquad \inf_{x\in X^{\mathcal{M}}} \psi^{\mathcal{M}}(x,y)$$

Remark 1.68. Some critical remarks:

- Zero-sets of definable predicates are uniform assignments
- If an assignment is a definable set then it is the assignment arising from the zero-set of some definable predicate. Just choose $\psi(x,y) = d(x,y)$ and parse $\inf_{x\in X^{\mathcal{M}}} \psi(x,y)$.
- "Definable sets are those you can quantify over" (in the discrete case, you can quantify over the solution set of any formula)
- A lot of zero sets are *not* definable sets. (examples later)

[7] I.e. when is the identity continuous at a point?

(terminology in the area in fuzzy: some use "definable" also for "definable predicate".)

Theorem 1.69. Suppose that $\mathcal{M} \mapsto X^{\mathcal{M}}$ is a uniform assignment realtive to a theory T. TFAE:

1. This assignment is a defianble set.

2. The distance predicate $d(x, X^{\mathcal{M}})$ is a definable predicate for T.

Proof.

$1 \Rightarrow 2$ Fix ψ. By uniform continuity there is a continuous α such that for all x, y, z we have $|\psi(x, z) - \psi(y, z)| \leq \alpha(d(x, y))$. Consider $\inf_z(\psi(x, z) + \alpha(d(z, X)))$ and[8] $\inf_{z \in X} \psi(x, z)$. We want to see that these are equal and the first is a definable predicate.

□

Theorem 1.70. Suppose that $\mathcal{M} \to X^{\mathcal{M}}$ is a uniform assignment. TFAE:

1. This is a deifnable set.

2. For all I and $\mathcal{U} \in \beta I$ and families of models of T, if $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ then $X^{\mathcal{M}} = \prod_{\mathcal{U}} X^{\mathcal{M}_i}$.

The above is the way that one checks definability in practice/applications. (recall that even without explicit realization in the $\mathcal{M}_i$ this can work out: it suffices to have approximations)

Example 1.71. Here are examples of definable sets:

- Finite products of sorts. Ranges of terms. It is clear that you can quantify over them. The ranges of definable functions are definable sets too.

- The ball of radius 1 around a point in the ball of radius 1 in an Hilbert space. More generally, if the underlying metric space has unique geodesics then balls will be definable sets.

- Ultrametrics dive non-definable examples. For instance on $[0, 2]$ define

$$d(x, y) = \begin{cases} \max\{x, y\} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

  If we fix 0 as a constant then the zero-set of $d(0, x) \leq 1$ does not survive ultrapowers. These type of examples arise naturally from metrics associated to certain valuations.

[8] Think of $\alpha(d(z, X))$ as a "penalty" for being outside X.

- There are lots of examples from functional analysis, but we will see them later.

Definition 1.72. A type is *principal* if the logic topology on the type space refines the metric topology on p.

Proposition 1.73. For a complete type p TFAE

1. p is principal
2. The zero set of p is definable
3. There are formulas φ_m and numbers $\delta_m > 0$ such that for every m, $p(\varphi_m) = 0$ and any complete type q, we have that if $\varphi_m(x) \leq \delta_m$ is in q then $d(p, q) \leq 1/m$.

($3 \Rightarrow 2$ comes from the ultraproduct characterization)

Theorem 1.74 (Omitting Types Theorem)**.** If $\mathcal{L}$ is a separable language, T is a complete theory in $\mathcal{L}$ and p is a compete type, then every model of T realizes p iff p is principal.

In this case it is very important that p is complete (in contrast with the discrete case).

Theorem 1.75. For a complete theory T in a separable language, T is separably categorical iff all complete types are principal.

1.5 Some Examples and Applications (14/06 - 15:00)

We'll skipping the section on imaginaries that is on the slides.

Let us see (an approximation of) the proof of the last theorem.

Proof Approximation. Take A and B and fix $a_0 \in A$. We want to find a $b_0 \in B$ to do back-and-forth. Let $p_0 = \text{tp}(a_0/\emptyset)$. By hypothesis there is a formula that tells us $d(x, z(p_0))$ (where z means "zeros"). Take the inf. Since the theory is complete the formula $\inf_x d(x, z(p_0))$ is also true in B. Take a small ε (say smaller than the diameter of B). (MISSING why there is b_0 realizing the inf, but there is.

What happens if we have constructed up to a_n and b_n and want to take care of some a?. Let $q = \text{tp}(\bar{a}_n a/\emptyset)$ (not $\text{tp}(a/\bar{a}_n)$ because we only have assumptions on types over the emptyset and, unlike in the discrete world, these two things are not the same: in other words if you name elements you can lose separable categoricity). The $\bar{a}_n$ realize $\inf_y d(x_0, \dots, x_n, y, z(q))$. Put $\bar{b}_n$ into that. We can find b such that $d(\bar{b}_n b, z(q)) < 2^{-n}$. In other words $\bar{b}_{n+1}$ will not extend $\bar{b}_n$. ("things are moving"). Thus at stage k we have $a_0^k, \dots, a_k^k$ and $b_0^k, \dots, b_k^k$ that do a 2^{-k}-approximation, and all the a_j^k (j

fixed, k varies) have the same type. After ω steps these are Cauchy sequences, so they converge by completeness. Extending this map by completeness completes the back-and-forth. □

Let us now see some example.

Example 1.76. That the language with d only, 1 sort and diameter 1. The following is a construction of a universal, separable metric space, the Urysohn space.

Recall these facts about finite metric spaces

Proposition 1.77 (Amalgamation)**.** If $f\colon A \to B$ and $g\colon A \to C$ are isometries and all spaces are finite then there is a finite D and isometries $h\colon B \to D$ and $j\colon C \to D$ such that $hf = jg$

Corollary 1.78. Since a 1-point space embeds into any metric space, we also have joint embedding.

Let A be a finite metric space and let $S(A)$ be the set of one-point extension of A. Topologize it with $L(A)$, the 1-Lipschitz maps on A, with this distance on $L(A)$:

$$d(f, g)0 \max_x |f(x) - g(x)|$$

Then $L(A)$ is a compact metric space. Identify $a \in A$ with $f_a\colon A \to [0, 1]$ by $f_a(x) = d(x, a)$. It can be checked that $a \mapsto f_a$ is an isometry. If $B = A \cup \{b\}$, a 1-point extension of A, identify b with $f_b(x) = d(x, b)$ for $x \in X$. One checks that $f_b \in L(A)$ and this is an isometric embedding over A. The space of 1-point extensions of A is a separable compact space.

Now we build a metric space $\mathcal{U}$ as the completion of the union of an increasing chain of finite spaces X_n (kind of a Fraïssé construction). For each finite $F \subseteq X_n$ we keep track of a countable dense subset of 1-point extensions of F and promise to amalgamate them all eventually. The inductive stage is something like: $F \subseteq X_n$ and G a 1-point extension of F. We let X_{n+1} be an amalgamation of X_n and G over F. Remember to take the completion at the end.

Theorem 1.79. The metric space $\mathcal{U}$ is separable complete and universal: it embeds isometrically all separable metric spaces. Moreover $\mathcal{U}$ is ultrahomogenous and its theory is separably categorical.

We will not see the proof, but let us identify some axioms that hold in $\mathcal{U}$ and one construction. If $\{a_1, \dots, a_n\}$ is an enumeration of an n-element metric space A then let $\mathrm{Con}_A(x_1, \dots, x_n)$, the configuration formula for A, be

$$\max\{|d(x_i, x_j)| - r_{ij} \mid i, j \leq n\}$$

where $r_{ij} = d(a_i, a_j)$. if $\mathrm{Con}(a_1, \dots, a_n) = 0$ and $\mathrm{Con}_A(b_1, \dots, b_n) = 0$ then the map $a_i \mapsto b_i$ is an isometry. What if $\mathrm{Con}_A(b_1, \dots, b_n) = 0 < \varepsilon$? MISSING

Take some space $B = \{b_1, \dots, b_n\}$ such that $\mathrm{Con}_A(b_1, \dots, b_n) = \Delta$. Build a metric space with elements $\{a_1, \dots, a_m, b_1, \dots, b_m\}$ with the given metrics on A and B. Set $d(a_i, b_i) = \Delta/2$ for all $i \leq n$ and for $i, j \leq n$

$$d(a_i, b_j) = \min\{d(a_i, a_k) + d(b_k, b_j) + \Delta/2 \mid k \leq n\}$$

It can be checked that this defines a metric on these points and the copy of A lies $\Delta/2$ away from the cop of B.

If we now have a one point extension $C = A \cup \{c\}$ of A, freely amalgamate B and C over A.

What is the value of $\mathrm{Con}_C(b_1, \dots, b_n, c)$? The interesting cases are $d(c, b_i)$ for $i \leq n$. But the value is Δ because MISSING

Thus given A any finite metric space and C a one point extension

$$\varphi_{A,C} 0 \sup |\mathrm{Con}_A(x_1, \dots, x_n) - \inf_y \mathrm{Con}_C(x_1, \dots, x_n, y)|$$

These sentences hold in Uryshon space and allow the back and forth argument that shows that its theory is separably categorical.

Here an some open questions:

Uryshon space feels like a Fraïssé construction and is in a technical sense. This makes it a generic object for the calss of finite metric space. Is it also in some sense a random object? More precisely, is Uryshon space elementarily equivalent to an ultraproduct of finite metric spaces?

Notice that its theory is very complicated: unstable, not simple... (if you do the same thing in the discrete case you jsut get an infinite set, which has a very simple theory.)

Let's now see another example. Fix an Hilbert space H and consider a linear operator A on H. We use the norm $\|A\| = \sup_x \frac{\|Ax\|}{\|x\|}$. Write $B(H)$ for the algebra of all bounded operators on H. This carries a natural complex vector space structure and multiplication is composition. There is an adjoint operation defined via the inner product on H: for $A \in B(H)$, A^* satisfied for all $x, y \in H$

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

The operator norm puts a normed linear structure on $B(H)$ and the norm satisfies the C^*-identity $\|A^* A\| = \|A\|^2$ for all $A \in B(H)$.

Definition 1.80. A concrete C^*-algebra is norm closed *-subalgebra of $B(H)$ for some Hilbert space H. An abstract C^*-algebra is a Banach *-algebra which satisfies the C^*-identity.

Example 1.81. For any Hilbert space H, $B(H)$ is a concrete C^*-algebra. In particular, $M_n(\mathbb{C})$, the $n \times n$ complex matrices is a C^*-algebra.

Example 1.82. $C(X, \mathbb{C})$, all continuous functions on a compact Hausdorff X is an abelian abstract C^*-algebra (adjoint is conjiugation, the norm is the sup-norm). These are all the unital ones by Gelfand and Naimark.

Example 1.83. C^*-algebras are closed as abstract C^*-algebras under direct sums and direct limits with $*$-homomorphism embeddings as connecting maps. Any finite dimensional C^*-algebra is the direct sum of finitely many copies of matrix algebras.

Theorem 1.84. Every abstract one is isomorphic to a concrete one.

Operator algebraists use ultraproducs: suppose A_i are C^*-algebras for all $i \in I$ and $\mathcal{U} \in \beta I$. Consider the bounded product

$$\prod^b A_i = \left\{ \bar{a} \in \prod A_i \;\middle|\; \lim_{\mathcal{U}} \|a_i\| < \infty \right\}$$

and the two-sided ideal $c_{\mathcal{U}} = \{\ldots \lim \|a_i\| = 0\}$. The ultraproduct is defined as the quotient.

In model theory we trat them as with Hilbert spaces: one sort per norm, inclusion maps between the balls, etc. The metric is given via $\|x-y\|$ on each ball. Uniform continuity holds (multiplication is an issue and that's why we have restricted the norm). The sorts are complete because C^*-algebras are. Is that an elementary class? C^*-algebras are closed under ultraproducts and subalgebras. (notice it is closed under ultraroots, too, but this says nothing about how to axiomatize it). Closure under subalgebras says that probably we should have an universal axiomatization[9].

Operator algebraists called "weakly stable relations" what we would call "quantifier free definable sets". The set of self-adjoints elements, projections, partial isometries, etc- are all definable and this is helpful to know when expressing certain results in continuous logic. Coupled with knowing that for a C^*-algebra A, $M_n(A)$ can also be viewed as a definable set, it is possible to access the "K-theory" of A. Understanding what fragment of the K-theory of an algebra is elementary is one open problem in the continuous model theory of C^*-algebras. There are both examples where the answer is "a lot" and where it is "not much".

Consider $A' \cap A^{\mathcal{U}} = \{b \in A^{\mathcal{U}} \mid [b, A] = 0\}$, where A is identified with the diagonal embedding.

Theorem 1.85. For any infinite-dimensional C^*-algebra A, if CH does nto hold then there are many non-isomorphic ultrapowers of A via ultrafilters on $\mathbb{N}$. Moreover, there are many non-isomorphic relative commutants as well.

[9] I'm not going to write them here, but they are on the slides. The tricky part is dealing with balls. The naive way of writing it uses an inf, but you can get around it introducing terms. It requires some functional analysis.

In other words, the theory is stable, but the theory of the relative commutant is unstable.

The ultraproduct construction is useful because of saturation. Relative commutants are to some extent saturated: quantifier-free saturated and sometimes more. Is there some abstract version of the relative commutant that could be useful in other contexts?

Chapter 2

Stability, Combinatorics, and NIP (Pierre Simon)

2.1 13/06 - 11:00

2.1.1 Pseudofinite

The goal is to study asymptotic properties of fine sets.

Take $(M_i \mid i \in I)$ a family of L-structures, $A_i \subseteq M_i$ finite subsets. Let $M_* = \prod_{\mathcal{U}} M_i \supseteq A_* = \prod_{\mathcal{U}} A_i$, for $\mathcal{U} \in \beta I$. How is A_* behaved? In finite combinatorics we're usually interested in counting things, so we're not really interest in A_* but in its density or something like that.

For each i, formula $\varphi(x; y)$ and $b \in M_i^{|y|}$, you can consider $m_i(\varphi(x; b_i)) = |\varphi(A_i; b_i)| \in \mathbb{N}$ and take the limit $m_*(\varphi(x; b)) = \prod_{\mathcal{U}} m_i(\varphi(x; b_i)) \in \mathbb{N}^{\mathcal{U}} \subseteq \mathbb{R}^{\mathcal{U}}$. This number usually carries too much information, and we need to compress it according to what we are interested in. We can extract various information from m_*, for instance

- If we're interested in "different scales", i.e. to compare sets of different size, we want to extract some "dimension" information. We will not see this.
- If we're comparing things "of the same scale" we want more of a "measure" thing. We will see this.

Definition 2.1. Fix a normalizing set $\psi(x)$. Define $\mu(\varphi(x; b))$ as

$$\mu(\varphi(x; b)) = \operatorname{st}\left(\frac{m_*(\varphi(x; b))}{m_*(\psi(x))}\right) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

were $\operatorname{st}(\cdot)$ is the usual standard part.

A special case of interest (which is what we will be concerned with) is when $\psi(x)$ is the formula "$x = x$"; then μ has values in $[0, 1]$.

Proposition 2.2. Another equivalent definition is

$$\mu(\varphi(x;b)) = \lim_{\mathcal{U}} \frac{|\varphi(A_i;b_i)|}{|A|_i}$$

2.1.2 Measures

Fix a complete T and a monster $\mathfrak{U} \models T$.

Definition 2.3. A *Keisler measure* over a set $A \subseteq \mathfrak{U}$ in variables x is a finitely additive probability measure the A-definable sets in variable x. That is, it satisfies

- $\mu(\varphi(x;a)) \in [0,1]$
- $\mu(x = x) = 1$
- $\mu(\varphi(x) \vee \psi(x)) + \mu(\varphi(x) \wedge \psi(x)) = \mu(\varphi(x)) + \mu(\psi(x))$
- $\mu(x \neq x) = 0$ (or maybe require $\mu(\neg\varphi) = 1 - \mu(\varphi)$.

Example 2.4. Here are some examples:

- A type $p(x) \in S_x(A)$ is a 2-valued measure.
- Convex combinations $\sum_{i<\omega} a_i p_i(x)$ (where $\sum_{i<\omega} a_i = 1$) of types (or, more generally, of measures).
- The pseudo-finite measure we constructed before.
- On $\mathbb{R}$, or maybe an o-minimal expansion (end extension) $R^* \succ (\mathbb{R}, +, \cdot, \ldots)$, we can take the Lebesgue measure on $[0,1]^* \subseteq R^*$ defined as $\mu(\varphi(x;b)) = \lambda(\mathrm{st}([0,1]^* \cap \varphi(x;b)))$, where λ is the classical Lebesgue measure. Equivalently, by o-minimality, $\varphi(x;b) \cap [0,1]$ is a finite union of disjoint intervals with endpoints a_i and b_i and we can take $\mu(\varphi(x;b)) = \mathrm{st}\left(\sum b_i - a_i\right)$.
- On the random graph (M, R) there is a natural measure defined as $\mu_x\left(\bigwedge_{i<n} (\neg) xR(a_i)\right) = 2^{-n}$, where the $(a_i)_{i<n}$ are pairwise distinct. So $\mu(xRa) = 1/2$, while $\mu(xRa \wedge \neg xRb) = 1/4$, etc. Notice that $\mu(x = a) = 0$. This is automorphism-invariant[1].

Definition 2.5. If μ_x is a measure over A, we define its *support* $S(\mu) \subseteq S_x(A)$ as $\{p(x) \mid \forall \varphi(x) \in p\ \mu(\varphi(x)) > 0\}$. Types in the support are sometimes called *weakly random for* μ.

Remark 2.6. $S_(\mu)$ is closed, almost by definition.

[1]Another one is: take the theory of an equivalence relation with two infinite classes and take the $1/2 - 1/2$ average of the two types. Notice that one of the two types will not be $\emptyset$-automorphism-invariant.

Given μ_x a measure over A, we can extend it uniquely to a regular Borel σ-additive measure on $S_x(A)$ (thing of the formulas as of the clopen sets of $S_x(A)$). We still call this measure μ. *Regular* means that for every Borel $X \subseteq S_x(A)$

$$\mu(x) = \sup_{\substack{C \subseteq X \\ C \text{ closed}}} \mu(C) = \inf_{\substack{O \supseteq X \\ O \text{ open}}} \mu(O)$$

We define it this way: if $O \subseteq S_x(A)$ is open and we write it a $O = \bigcup_{i \in I} D_i$, with the D_i clopen, we just set $\mu(O) = \sup_{I_0 \in \mathscr{P}_{\text{fin}}(I)} \mu\left(\bigcup_{i \in I_0} D_i\right)$. Then for $X \subseteq S_x(A)$ just define $\mu(X)$ by regularity. Conversely, from a regular measure on $S_x(A)$ you recover a measure on the clopen sets (hence on formulas) simply by restriction.

It is easy to check that

Proposition 2.7. $S(\mu)$ is the smalles closed subset of measure 1.

Definition 2.8. The set $\mathfrak{M}_x(A)$ of measures over A is topologized by its inclusion inside[2] $[0,1]^{\mathrm{Def}_x(A)}$ with the product topology.

With this topology it is compact, since it is closed in a compact space.

Example 2.9. Let's look at some supports:

- What is the support of the measure on the random graph we defined before[3]?. It is the set of non-realized types, because all $x = a$ formulas have measure 0.
- The support of a type p is $\{p\}$
- The support of an average of types $(p_i \mid i < omega)$ is the closure $\overline{\{p_i \mid i < \omega\}}$.
- The support of the Lebesgue measure on $[0,1] \subseteq R^*$ is the set of types $p \in S_x(R^*)$ such that
  - $p_x \vdash 0 \leq x \leq 1$, and
  - if $p_x \vdash a \leq x \leq b$ then there is $n \in \mathbb{N}$ such that $b - a \geq 1/n$.

Exercise 2.10. In the last example with the Lebesgue measure, $|S(\mu)| = 2^{\aleph_0}$.

Remark 2.11. This is in contrast to what happens with that measure on the random graph: there, if you model is very big, then the support is very big.

Question 2.12. When is it the case that all measures are convex combinations of types?

[2] $\mathrm{Def}_x(A)$ is the set of A-definable sets.

[3] Where $A = M$.

Recall that

Definition 2.13 (Assume L countable.[4]). T is *ω-stable* iff for all $M \vDash T$ we have that $S_x(M)$ has Cantor-Bendixson rank $< \infty$ (it can be an infinite ordinal).

Proposition 2.14. If T is ω-stable, then any $\mu \in \mathfrak{M}_x(A)$ can be written as $\sum_{i<\omega} a_i p_i$.

Proof. By hypothesis the Cantor-Bendixson rank of $S(\mu)$ is $< \infty$. Take its isolated points $\{p_i \mid i \in I\}$ and notice that for each i we have $\mu(\{p_i\}) > 0$, so there are at most $\aleph_0$ isolated points. Write

$$\mu = \sum_{i \in I} \mu(\{p_i\}) + \nu$$

and induct on[5] ν, which is possible by hypothesis. □

2.2 Discussion

Question 2.15. What about that "dimension" thing?

If you normalize by $x = x$ then $\mu(\varphi(x;b)) = \lim_{\mathcal{U}} \frac{|\varphi(A;b)|}{|A|_i}$. So you only see things that are linear in $|A_i|$ (write it like $|\varphi(A_i;b)| \sim c|A_i|$). What if you want to capture something that goes like $|\varphi(A_i;b)| \sim c|A_i|^\alpha$, i.e. to distinguish between things that have measure 0? So to capture α you just take logarithms.

The general construction of Hrushovski is: take a convex subgroup $C \subseteq \mathbb{R}^{\mathcal{U}}$, e.g. the convex hull of $\mathbb{R}$. Then define

$$d_C(\varphi(x;b)) = \log m_*(\varphi(x;b)) + C \in \mathbb{R}^{\mathcal{U}}/C$$

This captures the exponent destroying the constant (i.e. you know that something goes like $k\sqrt{|A_i|}$ but destroy k). This is called *fine dimension*. *Coarse dimension* is given normalizing so to have $\psi(x)$ of dimension 1 and letting C be the largest subgroup not containing $\log m_*(\psi(x))$ or something like that. If after this you do the above construction you can take the values to be real numbers. If you set $\psi(x)$ to be $x = x$ you exactly get α (after taking standard part). The coarse dimension can also be directly defined as

$$\operatorname{st}\left(\frac{\log(m_*\varphi(x,b))}{\log(m_*\psi(x))}\right)$$

Question 2.16. How do you find a non-average-of-types measure in a non-ω-stable theory?

[4] If I recall correctly otherwise this is the definition of *totally trascendental*.

[5] Possibly $\nu(x = x) < 1$, but it is not a problem.

Take a perfect set, cut it iteratively in two and then give it a measure as in the random graph (the level of the resulting tree with 2^n pieces will have each piece of size $1/2^n$). This measure will have no atoms.

2.3 14/06 - 09:00

2.3.1 Stable Bipartite Graphs

Definition 2.17. If $\varphi(x; y)$ is a formula, a φ-type over A is a maximal consistent set of instances of φ and $\neg\varphi$ with parameters in A. Call the space of such types $S_\varphi(A)$. A φ measure is a measure on Boolean combinations of instances of φ over A.

Proposition 2.18. If μ_x is some φ-measure over A then there are $p_i \in S_\varphi(A)$ such that μ is the average of them if and only if the following equivalent conditions hold:

1. $\varphi(x; y)$ is stable.
2. The Cantor-Bendixson rank of $S : \varphi(A)$ is less than ω.
3. The Cantor-Bendixson rank of $S : \varphi(A)$ is less than ∞.

Theorem 2.19 (Malliaris-Shelah, Malliaris-Pillay)**.** Let V, W, R be a bipartite graph, i.e. $R \subseteq V \times W$. Assure $R(x; y)$ is stable. Then given $\varepsilon > 0$ there is m such that for any finite $V' \subseteq V$ and $W' \subseteq W$ there are partitions $V' = \bigcup_{i<m} V_i$ and $W' = \bigcup_{i<m} W_i$ such that for any $(i, j) \in m^2$ then one of the following holds:

- (almost complete) For $(1 - \varepsilon)|V_i|$ vertexes $a \in V_i$ and $(1 - \varepsilon)|W_j|$ vertexes $b \in W_j$ we have $R(a, b)$, and dually reversing V_i and W_j.
- (almost empty) same, replacing R with $\neg R$.

Proof. Assume it is false, as witnessed by ε and, for each m, a counterexample $(V^{(m)}, W^{(m)})$. Consider $\prod_{\mathcal{U}}(V^{(m)}, W^{(m)}, R \restriction V^{(m)} \times W^{(m)}) = (V^*, W^*, R^*)$. Since stability is preserved by ultraproducts, $R^*(x, y)$ is stable. On $V^* \times W^*$ we have the pseudo-finite measures μ_x on V^* and λ_y on W^*. Apply the previous Proposition and write $\mu_x = \sum_{a_i p_i}$ for suitable $p_i \in S_R(W^*)$ and $\lambda_y = \sum b_i q_i$ for suitable $q_i \in S_{R^{\text{op}}}(V^*)$, where $R^{\text{op}}(y; x) = R(x; y)$.

Exercise 2.20. Complete the gap here in this proof (requires some stability theory).

We get definable $V^* = \bigcup_{i<m} V_i^*$ and $W^* = \bigcup_{i<m} W_i^*$ which has the required property. The same partition will work for $\mathcal{U}$-almost all factors in the ultraproduct, contradicting the assumption. □

2.3.2 NIP

Definition 2.21. Let $\mathcal{C} \subseteq \mathscr{P}(X)$. A subset $X_0 \subseteq X$ is *shattered* by $\mathcal{C}$ if $\mathcal{C} \cap X_0 = \mathscr{P}(X_0)$. We say that $\mathcal{C}$ has VC-dimension $< n$ if no set of size n is shattered by $\mathcal{C}$, and that it has infinite VC-dimension if for all n we have VC-dim$(\mathcal{C}) \geq n$.

Example 2.22. If $X = \mathbb{R}$ and $\mathcal{C} = \{(a, b) \mid a < b\}$, then VC-dim$(\mathcal{C}) = 2$.

Proof. It is easy to shatter any 2-point set, but if $x < y < z$ there is no way to isolate $\{x, z\}$ with an interval. □

Example 2.23. If $X = \mathbb{R}^2$ and $\mathcal{C}$ is the family of half-planes, VC-dim$(\mathcal{C}) = 3$.

In model theory we do not care about the exact value of the VC-dimension, we only care whether it is finite or not.

Assume that X is equipped with a probability measure μ and that there are some measurability condition on $\mathcal{C}$ (we will just care about the case where X is finite, so it is not important to see what these conditions are). On X^n we have the usual product measure.

Recall that (one version of) the law of large number say that if we take $Y \in \mathcal{C}$ and we take n points at random in X and look at how many of them land in Y this converges to $\mu(Y)$ as n goes to infinity. More precisely, fix $\varepsilon > 0$ and let $(X_i)_{i<\omega}$ be independent identically distributed, then

$$\mu^n \left(\{(X_1, \ldots, X_n\} \;\middle|\; \frac{1}{n}|\{X_1, \ldots, X_n\} \cap Y| - \mu(Y) \geq \varepsilon \right) \xrightarrow{n\to\infty} 0$$

The VC-theorem is a kind of *uniform* law of large numbers.

Theorem 2.24 (VC (Vapnik-Chervonenkis))**.** If $\mathcal{C}$ has finite VC-dimension, then

$$\mu^n \left(\{(X_1, \ldots, X_n\} \;\middle|\; \exists Y \in \varphi \, \frac{1}{n}|\{X_1, \ldots, X_n\} \cap Y| - \mu(Y) \geq \varepsilon \right) \xrightarrow{n\to\infty} 0$$

The speed of convergence is exponential and only depends on the VC-dimension.

Exercise 2.25. Equispaced points random in the circle, where $\mathcal{C}$ is the family of intervals.

Corollary 2.26. Given d and $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that for any X and $\mathcal{C}$ such that VC-dim$(\mathcal{C}) \leq d$ there are $(x_1, \ldots, x_n) \in X^n$ such that for all $Y \in \mathcal{C}$

$$\left| \frac{1}{n}|\{X_1, \ldots, X_n\} \cap Y| - \mu(Y) \right| \leq \varepsilon$$

Definition 2.27. Let (V, W, R) be a bipartite graph. We say that $R(x; y)$ is NIP if $\{R(a; W) \mid a \in V\}$ has finite VC-dimension.

Remark 2.28. This is actually symmetric, i.e. equivalent to asking finite VC-dimension for $\{R(V;b) \mid b \in W\}$.

Definition 2.29. A formula $\varphi(x,y;t)$ is *rectangular*[6] if it is a boolean combination of rectangles $\theta_1(x;t) \wedge \theta_2(y;t)$.

The following says that "NIP bipartite graphs are very close to be rectangular".

Theorem 2.30. Let (V,W,R) be a bipartite graph, with R NIP. Given $\varepsilon > 0$, there is a rectangular formula $\varphi(x,y;t)$ such that for any finite $V' \subseteq V$ and $W' \subseteq W$ there are parameters b such that

$$|(R \restriction V' \times W') \triangle \varphi(V',W',b)| \leq \varepsilon |V'| \cdot |W'|$$

Remark 2.31. Rectangular formulas in dimension 1 this case are Boolean combinations of subsets of V' (or W').

2.4 14/06 - 13:30

2.4.1 Combinatorics

Le us prove the previous theorem.

Proof. Pick finite $V' \subseteq V$, $W' \subseteq W$. Let $\psi(x;y_1,y_2) = R(x;y_1) \triangle R(x;y_2) \subseteq V \times W^2$. This is NIP because finite combinations of NIP are NIP. Consider $X = V'$ with the normalized counting measure and set $\mathcal{C} = \{\psi(V';b_1,b_2), b_1, b_2 \in W'\}$, which has finite VC-dimension by NIP. (notice that ψ does not depend on V' or W', so this will give a uniform result). Let $x_1,\dots,x_n \in V'$ be given by the VC-theorem. Partition $W = \bigsqcup_{i<m} W_i$ in a way such that for every i and $b_1, b_2 \in W_i$, we have $\mathrm{tp}_R(b_1/\{x_1,\dots,x_n\}) = \mathrm{tp}_R(b_2/\{x_1,\dots,x_n\})$. Notice that[7] $m \leq 2^n$. For each i, pick $b_i \in W_i$ and partition $V' = \bigsqcup_{j<k}$ in the same way, i.e. such that $\mathrm{tp}_R(a_1/\{b_1,\dots,b_n\}) = \mathrm{tp}_R(a_2/\{b_1,\dots,b_n\})$. Notice that $|R(V',b_1) \triangle R(V',b_2)| \leq \varepsilon |V'|$ by construction (that symmetric difference does not contain any x_i and b_1 and b_2 have the same type over the x_i).

Consider the rectangular formula corresponding to the union[8]

$$\bigcup_{\substack{(i,j) \\ \forall a \in V_i\ R(a,b_j)}} V_i \times W_j$$

$\square$

[6] In x, y, while t is a variable where parameters will be put in.
[7] Actually it is polynomial.
[8] This does not depend on a because all of them have the same type over the b_j.

2.4.2 Back to Model Theory

Definition 2.32. Fix T and $M \vDash T$. We call $\varphi(x;y)$ NIP if $\{\varphi(M;b) \mid b \in M^{|y|}\}$ has finite VC-dimension.

Remark 2.33. This does not depend on M and is symmetric.

Example 2.34. If $\leq$ is a linear order, $x \leq y$ is NIP.

Definition 2.35. T is NIP if all formulas are NIP.

Example 2.36.

- All stable theories (e.g. Th($\mathbb{C}$)) are NIP.
- DLO is NIP.
- Any o-minimal theory is NIP (e.g. Th($\mathbb{R}$) = RCF).
- $\mathbb{Q}_p$, $\mathbb{Q}_p((t))$.
- ACVF.

Essentially the main examples are stable, o-minimal, and valued fields.

Theorem 2.37. If all formulas[9] $\varphi(x;\bar{y})$ with $|x| = 1$ are NIP, then T is NIP.

Example 2.38.

- The random graph is not NIP.
- Pseudofinite fields are not NIP.
- Set theory, arithmetic, almost any theory.

2.4.3 Measures in NIP

The VC-theorem has the following corollary:

Corollary 2.39 (T NIP[10])**.** Let μ_x be a measure over some A, fix a formula $\varphi(x;y)$ and $\varepsilon > 0$. Then there are $p_1, \ldots, p_n \in S_x(A)$ such that if $\mu_0 = \frac{1}{n}\sum p_i$ then for all $b \in A$

$$|\mu(\varphi(x;b)) - \mu_0(\varphi(x;b))| \leq \varepsilon$$

Corollary 2.40 (L countable[11])**.** There is a countable $S_0 \subseteq S_x(A)$ such that $S(\mu) \subseteq \overline{S_0}$.

Proof. Let φ vary and let $\varepsilon \to 0$. □

Corollary 2.41. $|S(\mu)| \leq 2^{2^{\aleph_0}}$ (actually $\leq 2^{\aleph_0}$ but we will not see why).

Recall that we saw that this was true for the Lebesgue measure in RCF. It was false for the average measure on the random graph[12].

[9] Suppose that we are working in a 1-sorted structure.
[10] Maybe most will work for a NIP formula in a IP theory.
[11] But A can be as big as you want
[12] Probably the same thing works for any IP theory.

Due to these approximation results, the general philosophy in NIP is "treat measures as types".

What happens when we take a pseudofinite measure in NIP context? Fix T NIP, $M_i \models T$, finite sets $A_i \subseteq M_i$, and on $M^* = \prod_{\mathcal{U}} M_i$ we have the pseudofinite measure $\mu_x = \lim_{\mathcal{U}} m_i$ coming from the normalized counting measures m_i on A_i. What does the VC-theorem give us?

$$\forall i \, \forall \varphi(x;y) \, \forall \varepsilon > 0 \, \exists a_1^i, \ldots, a_n^i \in A_i \, \forall b \in M_i \left| \frac{1}{n} |\{a_1^i, \ldots, a_n^i\} \cap \varphi(x;b)| - m_i \varphi(x;b) \right| \leq \varepsilon$$

Notice that n only depends on φ and ε. In the ultraproduct, this is still true when setting, say, $a_1 = [a_1^i]$. I.e.

$$\forall b \in M^* \left| \frac{1}{n} |\{a_1, \ldots, a_n\} \cap \varphi(x;b)| - \mu(\varphi(x;b)) \right| \leq \varepsilon$$

This is stronger then the previous approximation result because now we have realized types, i.e. points.

Definition 2.42. A measure that can be approximated with *points* ($\forall \varphi \forall \varepsilon \; \exists a_1, \ldots, a_n \ldots$) is called *generically stable*-

This means that this measure has very nice properties.

Proposition 2.43. Let $\mu_x \in \mathfrak{M}(M)$ be generically stable; then

1. μ is *definable*[13]: $\forall \varphi(x;y); \forall r_1 < r_2 \exists \theta(y;d) \in L(M)$ such that $\{b \mid \mu(\varphi(x;b)) \leq r_1\} \subseteq \theta(M;d) \subseteq \{b \mid \mu(\varphi(x;b)) \leq r_2\}$. This is true because d will be the a_i and θ will say "you are between those two proportions of the a_is".

2. For any $N \succ M$ there is an extension of μ to $\tilde{\mu} \in \mathfrak{M}(N)$ which is also generically stable defined by the same formulas. You can see this by definability or (which actually is the same, given our proof of definability) by doing the average on the a_is in the bigger model too. In particular μ extends canonically[14] to a measure over the monster $\mathfrak{U}$, which is M-invariant, i.e. invariant under $\mathrm{Aut}(\mathfrak{U}/M)$ (the definition is invariant because only depends on the a_is, which are in M).

3. $\tilde{\mu}$ is finitely satisfiable in M, i.e. every $\varphi(x;b)$ such that $\tilde{\mu}(\varphi(x;b)) > 0$ has a point in M (just take ε small enough).

Proposition 2.44 (T NIP)**.** Let $\tilde{\mu} \in \mathfrak{M}_x(\mathfrak{U})$ and assume that for some small $M \prec \mathfrak{U}$ the measure $\tilde{\mu}$ is definable and finitely satisfiable in M. Then $\tilde{\mu}$ is generically stable.

[13] A generalization of the notion of definable types.

[14] It can be proven that this does not depend on the a_is (in this particular case it is even unique).

2.5 15/06 - 09:00

2.5.1 Generically Stable Measures

Assume T NIP.

Example 2.45. Some examples of non-generically stable measures:

- Every pseudofinite measure is generically stable
- If T is stable, all measures are generically stable
- The Lebesgue measure is generically stable (proof MISSING, but see below)

Theorem 2.46. Let T be NIP and $M \models T$. Assume that M is equipped with a σ-algebra Σ such that[15] for any $N \succ M$ we have that for all $\varphi(x;b)$, with $b \in N$, we have that $\varphi(M^2;b) \subseteq M^2$ is Σ^2-measurable. Let μ_0 be a σ-additive measure on (M,Σ). Then this defines a Keisler measure μ on M by[16] $\mu(\varphi(x;b)) = \mu_0(\varphi(M,b))$. Then μ is generically stable.

The proof uses the VC-theorem to get approximation by points.

Example 2.47. This applies for instance to $\mathbb{R}$ and $\mathbb{Q}_p$.

There is a characterization of generically stable measures among definable measures: they are the only one that satisfy Fubini; if μ_x and λ_y are two definable measures one can define $\mu_x \otimes \lambda_y(\varphi(x,y)) = \int \mu(\varphi(x;b))\,\mathrm{d}\lambda_b$.

Proposition 2.48. If μ_x is definable then μ is generically stable iff $\mu_{x_1} \otimes \mu_{x_2} = \mu_{x_2} \otimes \mu_{x_1}$.

We will not go into this. Anyway this is how the previous theorem is proved (we know that σ-additive measures satisfy Fubini).

2.5.2 Distality

Consider the class of NIP structure. Inside it there is the subclass of stable structures. Some theories, like DLO or RCF are very far from stable. The idea of distality is to capture the "purely instable" or "order-like" NIP structures.

If, in a sense, "stable" is "sort of like algebraic geometry", then "distal" is "sort of semialgebraic geometry over $\mathbb{R}$ or $\mathbb{Q}_p$". O-minimal theories are distal. Anyway $\mathbb{Q}_p$ is not o-minimal. "Distal", unless "o-minimal", does not assume you have something (an order) in the language.

[15] This will imply that every externally definable set is measurable.
[16] Here $b \in M$.

Typically, in analysis (or semialgebraic geometry), you have a continuous function and, given some $\varepsilon > 0$, you want to cut the domain in small intervals such as on each of them the function moves at most ε. The point is that if you take a smaller ε, you take smaller intervals, but *they're still intervals*, i.e. the defining formula is the same.

Definition 2.49. Let $\varphi(x; y)$ be NIP. $\varphi(x; y)$ is distal if there is $\psi(x; z)$ such that for any finite $A \subseteq M^{|y|}$, we can write $M^{|x|} = \bigcup_{i<n} \psi(x; b_i)$ (we call these instances of ψ *cells*; they needn't be disjoint) such that for any $i \le n$ and $a, a' \in \psi(x; b_i)$ we have that $\mathrm{tp}_\varphi(a/A) = \mathrm{tp}_\varphi(a'/A)$.

In the previous example φ would be $y_1 \le f(x) \le y_2$ (or maybe $f(x) \le y$) and ψ would be $z_1 \le x \le z_2$.

Remark 2.50. NIP implies that you can find the b_is in A. If you assume it, NIP will follow.

Definition 2.51. T is distal if all formulas are distal.

Example 2.52. $\mathbb{R}$, o-minimal, $\mathbb{Q}_p$, $\mathbb{Q}_p((t))$ are distal. Stable structures are not, unless they are finite[17].

There are many characterizations of "distal". Even with "distal", it suffices to check in dimension 1:

Theorem 2.53 (T NIP)**.** If all formulas $\varphi(x; y)$ with $|x| = 1$ are distal, then T is distal. Same with $|y| = 1$.

One of the two facts in the previous theorem is nontrivial. (Question: what about them both together? Answer: probably not, just take some language where there is only structure in dimension 3 and above, e.g. in the language there is only a ternary relation symbol)

Theorem 2.54 (Chernikov-Starchenko)**.** Let (V, W, R) be definable in a NIP structure M, assume that $R(x, y)$ is distal in M and let $\varepsilon > 0$. There are rectangular formulas $\varphi_1(x, y; t)$, $\varphi_2(x, y; t)$ such that for any finite $V' \subseteq V$, $W' \subseteq W$ there is d such that

$$\varphi_1(V', W'; d) \subseteq R \restriction V' \times W' \subseteq \varphi_2(V', W'; d)$$

and $|\varphi_2(V', W'; d) - \varphi_1(V', W'; d)| \le \varepsilon |V'||W'|$.

It follows that if you graph is very big it will contain complete bipartite graphs. This is false in stable theories (probably with the graph of equality in the theory of equality).

[17] "Is it obvious?" "Depends on what you call obvious. It's an exercise."

Proof. Pick V', W' and $\varepsilon > 0$. Similarly to the proof in the general NIP case, consider

$$\zeta(y; z, x_1) \equiv \underbrace{\forall x(\psi(x; z) \to R(x; y))}_{\theta(y;z)} \triangle R(x_1; y)$$

Notice that, trivially, $\theta(y; d) \wedge \psi(x; d) \to R(x; y)$. Since we are in a NIP theory, ζ is NIP[18]. By the VC-theorem find $b_1, \ldots, b_n \in W'$ such that the average $\frac{1}{n}\sum_{i<n}(y = b_1)$ approximates the measure of every instance of ζ up to ε. $\zeta(y; d, a) \subseteq W'$. By distality, we can cover V' as a finite union of instances of ψ, say $V' = \bigcup_{i<m} \psi(x; d_i)$ such that

$$\forall a, a' \vdash \psi(x; d_1) \;\; \mathrm{tp}_R(a/\{b_1, \ldots, b_m\}) = \mathrm{tp}_R(a'/\{b_1, \ldots, b_m\})$$

Take now $\bigcup \theta(y; d_i) \wedge \psi(x; d_i)$, which is included in $R(x; y)$. We claim that this is big enough. Take $a \vDash \psi(x; d_i)$. Then $\theta(y; d_i) \cap \{b_1, \ldots, b_n\} = R(a; y) \cap \{b_1, \ldots, b_n\}$. Hence $|\theta(W', d_i) \triangle R(a; W')| \leq \varepsilon |W'|$ and by Fubini (or double-counting) we're done. Then do the same thing with $\neg R$ to get the other formula. □

The same results work as well if, instead of taking finite things and counting measures, you take generically stable measures (with the same proofs).

[18] And since it contains quantifiers we need the full theory to be NIP, not just R.

Chapter 3

Descriptive Set Theory and Model Theory (Phillip Wesolek)

3.1 Polish Groups and Ample Generics (15/06 - 10:30)

3.1.1 Ample Generics

Definition 3.1. A polish group is a topological group such that the topology is separable and completely metrizable.

Definition 3.2. A polish group G has *ample generics* if for all n the diagonal conjugacy action $g \cdot (h_1, \ldots, h_n) = (gh_1g^{-1}, \ldots, gh_ng^{-1})$ of G on G^n has a comeagre[1] orbit.

Example 3.3. Let us see some examples of groups that have ample generics:

- $S_\infty = \mathrm{Sym}(\mathbb{N})$ has ample generics (give $\mathbb{N}$ the discrete topology and then consider the pointwise convergence topology, in other words see it inside $\mathbb{N}^\mathbb{N}$), where the element with a comeagre orbit is given by, for all n, infinitely many disjoint n-cycles. (Exercise)
- (Hrushovski) The automorphisms of the random graph
- (Rwiatkorwiska)The homeomorphisms of the cantor space
- (Solecki) The isometry group of the rational Uryshon space.
- $\mathrm{Aut}(\mathbb{Q}, <)$ has *not* ample generics (Truss)

[1] A set is meagre if it is a countable union of nowhere dense set.

Question 3.4 (Truss)**.** Is there a characterization of when[2] Flim K has ample generics in terms of K? The answer is yes (Kochric-Rosendal).

Let us see some applications of ample generics. G will denote a polish group.

Theorem 3.5 (Kechris-Rosendal)**.** Say G has ample generics and $\{f_i A k_i\}_{i\in\mathbb{N}}$ is a cover of G, with $A \subseteq G$. Then $(A^{-1}AA^{-1}A^{-1}A)(A^{-1}AAA^{-1}A)$ is[3] a neighbourhood of 1.

Corollary 3.6. If G has ample generics, then G has ACP.

Theorem 3.7 (MISSING)**.** There exist nontrivial connected groups with ample generics.

Example 3.8. $L^0([0,1],\lambda,S_\infty)$.

Theorem 3.9 (W.)**.** No nontrivial locally compact polish group has a comeagre conjugacy class.

Question 3.10. Is there a non-trivial CLI[4] group with a comeagre conjugacy class?

3.1.2 Group Theory

Definition 3.11. An action of G on X (polish group on MISSING topological space) is topologically transitive if for all open $W, V \subseteq X$ exists $g \in G$ such that $gW \cap V \neq \emptyset$.

Theorem 3.12 (Main technical theorem for ample generics)**.** Let $G \curvearrowright X$. Then there is a comeagre orbit iff

1. The action is topologically transitive (this ensures a dense orbit)
2. For any open $\emptyset \neq V \subseteq X$ and $U \ni 1$ neighbourhood of 1 there is $\emptyset \neq W \subseteq V$ opensuch tat $U \curvearrowright W$ is topologically transitive. (this ensures there is a non-meagre orbit[5])

Proposition 3.13. TFAE for $G \curvearrowright X$:

1. it is topologically transitive
2. There is a dense orbit
3. There is a comeagre set of elements with a dense orbit

[2] F stands for Fraïssé.

[3] The parentheses are there just to emphasize that RHS is the inverse of LHS.

[4] A class bigger than "locally compact", somehow connected to model theory of countable structures.

[5] A dense non-meagre orbitis comeagre.

Proof.

(2 ⇒ 1) Clear

(3 ⇒ 2) Obvious

(1 ⇒ 3) Fix a countable base $(U_n)_{n\in\mathbb{N}}$ for the topology on X. For each n the set

$$GU_n = \bigcup_{g\in G} gU_n$$

is a dense open set, hence comeagre. By the Baire Category Theorem (or maybe you do not need it), $E = \bigcap_{n\in\mathbb{N}} GU_n$ is still open dense, hence comeagre and each $x \in E$ has a dense orbit.

□

Lemma 3.14 (Main technical lemma)**.** Suppose $G \curvearrowright X$. TFAE

1. There is a non-meagre orbit Gx
2. There is a non-empty open set $O \subseteq X$ with the following property: $\forall \emptyset \neq V \subseteq O$ and neighbourhoods U of 1 in G there is an open $\emptyset \neq W \subseteq V$ such that $U \curvearrowright W$ is topologically transitive.

If 1 holds then $O = U(Gx)$ (the largest open set in which Gx is comeagre)

Proof of the Theorem. Suppose Gx is comeagre. In particular gx is WHAT in X. Applying the Proposition, we deduce 1. Applying the Lemma, we deduce 2. Conversely, the Proposition ensures $\exists x \in X$ such that Gx is dense. Indeed, there is a comeagre set of such x. By the Lemma 2 implies that there is a non-meagre orbit Gy. Since it is non-meagre, it intersects the collection of $x \in X$ with dense orbit. Therefore Gy is dense and non-meagre, whereby (exercise) is comeagre. □

3.2 Automorphism Groups (15/06 - 15:00)

Reference: Kechris, Rosendal: Turbulence, amalgamation, and generic automorphisms.

$\mathcal{L}$ will be a countable first-order language and $\mathcal{K}$ will be a clas of finitely generated $\mathcal{L}$-structures.

Definition 3.15. $\mathcal{K}$ is said to have the hereditary property HP if when $A \in \mathcal{K}$ and B is a finitely generated $\mathcal{L}$-structure embeddable in A, then $B \in \mathcal{K}$. K has the joint embedding property JEP if any two structures of $\mathbb{K}$ can be embedded in a common structure.

(TOO FAST, THERE ARE SLIDES)

Language for metric spaces with distances in $\mathbb{Q}$: one predicate symbol for each distance.

The $_p$ is probably for "partial".

Typo on slide 11: embeddings are not necessarily isomorphisms.

Slide 13: the "inner diamond" (withouth i) does not necessarily commute.

It is a good exercise to work these things out for finite sets (limit $\mathbb{N}$, no structure)

3.3 Polish Groups and Coarse Geometry (16/06 - 10:30)

(there are slides, just taking small notes) Finitely generated groups have a geometry (i.e. a unique quasi-isometry type). The goal is finding something like that for Polish groups. This will be a *coarse type*, but for certain special Polish groups there is a quasi-isometry type.

Good exercise: proof what's on slide 10 about coarsely equivalent metrics. Coarse equivalence, quasi-isometry etc can be characterized in terms of asymptotic cones.

Exercise 3.16. Uniformly equivalent metrics give the same uniformity.

Example 3.17. Compact sets are coarsely bounded.

The point of the main theorem is that a coarse structure is given by a family of metric, but we would like to have just one.

3.4 Geometry of Automorphism Groups (16/06 - 13:30)

Geometric group theorists study the geometric structure (quasi-isometry classes) of finitely generated groups.

Definition 3.18. A set $A \subseteq G$ is *coarsely bounded* if $\forall V \ni 1$ open there is a finite $F \subseteq G$ and $k \geq 1$ such that $A \subseteq (FV)^k$.

Fact 3.19. If G is locally coarsely bounded then G admits a coarsely proper metric.

Fact 3.20. If G is generated by a coarsely bounded set, then G has a unique up to quasi-isometry coarsely proper metric.

Definition 3.21. A map $\varphi\colon (X, d_x) \to (Y, d_y)$ between metric spaces is a quasi-isometry if there are K and C such that for all $x, y \in X$

$$\frac{1}{k} d_x(x, y) - C \leq d_Y(\varphi(x), \varphi(y)) \leq K d_x(x, y) + C$$

and $\sup_{y \in Y} d_Y(ym\varphi(x)) < \infty$.

(slide 5): if you do not mind changing the language you can add predicates for the orbits instead of assuming ω-homogeneity.

(slide 6) The need for orbital graphs comes from the fact that if you are in, say, the compactly generated case, Cayley graphs can become continuum-branching and you don't want to look at that.

References: Geometry of Polish Groups, (another paper I did not got the name: blah automorphisms blah)

Chapter 4

Topological Dynamics and Model Theory (Krzysztof Krupiński)

4.1 15/06 - 13:30

The idea is to use topological dynamics to extend stable group theory to unstable contexts. Ben Yaacov and others also have an approach involving functional analysis, but we will not see this.

Definition 4.1. Let G be a topological group. A G-flow is a pair (G, X) where X is a compact Hausdorff space on which acts continuously.

Definition 4.2. A *discrete* flow is a G-flow for G equipped with the discrete topology.

In other words in this case this is just an action by homeomorphisms. In the general case we want the action to be continuous as a map $G \times X \to X$.

Definition 4.3. A G-ambit is a triple (G, X, x_0) where (G, X) is a G-flow and the orbit Gx_0 is dense.

Definition 4.4. (G, X) is a minimal flow if it has no proper subflows, i.e. no closed Y such that $GY \subseteq Y$.

These are important in topological dynamics; for instance some flows called *distal* are always disjoint unions of minimal flows, so if you understand them. . .

Remark 4.5. Each flow[1] contains a minimal subflow.

Proof. Compactness and Zorn's Lemma. □

[1] I.e. G-flow for some G.

Example 4.6 (Bernoulli shift)**.** Let $G = \mathbb{Z}$ and $X = 2^{\mathbb{Z}}$ with the product topology (i.e. the pointwise convergence one) and the action given by $(k \cdot f)(n) = f(n-k)$.

Remark 4.7. There is $f_0 \in 2^{\mathbb{Z}}$ such that $\mathbb{Z}f_0$ is dense in $2^{\mathbb{Z}}$. It is enough to take f_0 containing all finite sequences of $\{0, 1\}$. So $(\mathbb{Z}, 2^{\mathbb{Z}}, f_0)$ is a $\mathbb{Z}$-ambit.

If $f \in 2^{\mathbb{Z}}$ is periodic, $\mathbb{Z}f$ is finite, hence closed, and it is a minimal subflow. Anyway

Exercise 4.8. There are other, infinite, minimal subflows of $2^{\mathbb{Z}}$.

Definition 4.9. (G, X, x_0) is a universal G-ambit if for every ambit (G, Y, y_0) there is $h\colon (X, x_0) \to (Y, y_0)$ which is an homomorphism of G-ambit, i.e. a continuous $X \to Y$ such that $x_0 \mapsto y_0$ and preserving the action of G. Similarly for the notion of an universal minimal G-flow.

Remark 4.10. Such homomorphisms are automatically onto, by density in the first case and by minimality in the second one.

There are dozens of papers where people just compute universal minimal flows for some topological groups.

Fact 4.11. For every G there exists a universal G-ambit (minimal G-flow).

Proof. Let $\{(G, X_i, x_i) \mid i \in I\}$ be the collection of all G-ambits up to isomorphism (there are only a bounder number of possibilities, something like $2^{2^{|G|}}$). Then $\prod_{i \in I} X_i$ is a G flow, and (G, X, x) is universal, where $x = (x_i)_{i \in I}$ and $X = \overline{G \cdot (x_i)_{i \in I}}$. This is universal because you just need to consider the projection on the appropriate coordinate. □

Fact 4.12. Universal G-ambits are unique up to isomorphisms. Same thing for universal minimal G-flows.

Proof. Usual universality argument: take two of them and compose the relevant maps. Then use the fact that there is a fixed point and that since the action is continuous... For universal minimal G-flows it is more complicated. □

Example 4.13.

- If G is a compact group (G, G, e) is the universal G-ambit and (G, G) is the universal minimal G-flow.
- If G is discrete[2] then $(G, \beta G, p_e)$ is the universal G-ambit.

[2] Otherwise you have to use something like the *Samuel compactification*.

Idea of the proof. Consider the map $G \to \beta G$ that maps $g \mapsto g \cdot p_e = p_g$. Then take an aribtrary G-ambit (X, x_0) and consider the map $\varphi\colon G \to X$ defined by $g \mapsto g \cdot x_0$. We want to construct $f\colon (\beta G, p_e) \to (X, x_0)$ that commutes with the other two. If $p \in \beta G$, consider $\bigcup_{U \in p} \overline{\varphi[U]}$. This intersection is nonempty and it can be shown that it consists of just one point, that we will use as $f(p)$. (We used discreteness in having the embedding of G into βG continuous) □

(we're not going to see it, but with model theory you can describe something like the Samuel compactification as a quotient of a model by a type-definable equivalence relation)

- $\mathrm{Aut}(\mathbb{Q}, <)$ and $H_+([0,1])$ (non-decreasing homeomorphisms) are extremely amenable, i.e. the universal minimal flows are trivial (the second follows from the first, but the proof for the first need some Ramsey theory)
- The universal minimimal flows of $H_+(S^1)$ (order preserving homeomorphisms of S^1) is the natural action of $H_+(S^1)$ on S^1. This follows from extreme amenability of $H_+([0,1])$, because stabilizers of points are isomorphic to $H_+([0,1])$.

Definition 4.14. Let (G, X) be a G-flow and let $E(X) = \overline{\{\pi_g \mid g \in G\}} \subseteq X^X$, with the product topology, where the $\pi_g\colon X \to X$ are the maps $\pi_g(x) = gx$. Consider $(E(X), \circ)$ (exercise: show that $E(X)$ is actually closed under $\circ$). This is called the *Ellis semigroup of* (G, X).

Fact 4.15. In $(E(X), \circ)$

- $\circ$ is continuous on the left[3], i.e. for all f the map $\eta \mapsto \eta \circ f$ is continuous (the topology is the one of pointswise convergence!
- $E(X)$ is compact Hausdorff.

Exercise 4.16. $E(X)$ is a G-flow and $E(X) \cong E(E(X))$ both as G-flows and as semigroups.

Hint. The isomorphism is $f \mapsto f \circ -$. □

Fact 4.17 (Ellis)**.** Let S be a compact Hausdorff semigroup such that the semigroup operation is continuous on the left. Let M be a minimal left ideal in S (i.e. $S \cdot M \subseteq M$). Then

1. for all $p \in M$ we have that $Sp = Mp = M$
2. M can be written as the disjoint union $\bigsqcup_{u = u^2 \in M} uM$

[3] $\circ$ need not be continuous on the right (pointwise limits of continuous functions need not be continuous, so in $E(X)$ there may be discontinuous functions)

3. For every $u = u^2 \in M$, uM is a group.

4. For every $u, v \in \mathrm{Idem}(M)$ (i.e. the idempotents $\{p^2 = p\}$) we have $uM \cong vM$. Moreover the isomorphism type[4] of uM does not depend on M.

Definition 4.18. The *Ellis group* of the flow (G, X) is the group uM for any minimal left ideal M of $E(X)$ and $u \in \mathrm{Idem}(M)$.

In topological dynamics you can explain many dynamical properties of X in terms of $E(X)$. For instance X is distal iff $E(X)$ is a group.

Example 4.19.

- If G is a compact group $E(G, G) = G$, and it is (trivially) already a minimal ideal and its Ellis group.

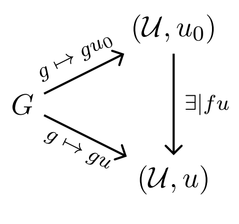
- (more on this tomorrow) if G is definable *in a stable structure* M and G acts on $S_G(M)$. This is universal in a certain category of definable ambits and we have a semigroup operation on $S_G(M)$ that makes it isomorphic to $E(S_G(M))$. The operation is given by: a realization of $p * q$ is the product of some $a \models p$ and some $b \models q$ such that $a \mathop{\smile\!\!\!\!\!\!|}_M b$. This has a unique minimal left ideal, and it is also a group and is the collection of all generic types (types that contain only generic sets, where a set is generic if finitely many translates of it cover G). Also, this does not depend on the model.

4.2 16/06 - 09:00

Exercise 4.20. Minimal G-subflows of $E(X)$ are exactly the minimal left ideals of $E(X)$.

Fact 4.21. All minimal G-subflows of $E(X)$ are isomorphic as G-flows.

Let $(G, \mathcal{U}, u_0)$ be the universal G-ambit.



Define $\mathcal{U} \ni p * u = f_u(p)$. It can be easily shown that

- $*$ is continuous on the left,

[4]As abstract groups, but there is something called τ-topology (T1, the operation becomes *separately* continuous in the two coordinates) under which more things happen.

- $*$ extends the action of G, i.e. $\pi(g) * u = (gu_0) * u = gu$,
- $*$ is associative (this is less obvious; exercise).

So from $\ell_p \colon \mathcal{U} \to \mathcal{U}$ given by $\ell_p(u) = p * u$ we have a map $\ell \colon \mathcal{U} \to E(\mathcal{U})$. One can check that ℓ is an isomorphism of G-flows and of semigroups. (notice that it depends on u_0, and that it will go into the identity of $E(\mathcal{U})$). This allows us to use the semigroup operation on $\mathcal{U}$ instead of going into the function space $E(\mathcal{U})$. If G is discrete, then we have $*$ on βG, because we know that it is the universal G-ambit.

Let now M be a first order structure, $\mathfrak{U}$ a monster for M, $G = G(M)$ a $\emptyset$-definable group and $G^* = G(\mathfrak{U})$ and $N \succ M$ an $|M|^+$-saturated model (I'll write $M \prec^+ N$). See $S_G(M)$ as the space of ultrafilters on the definable subsets of G. We will denote $S_{G,\text{ext}}(M)$ the space of ultrafilters on the externally definable[5] subsets of G. We can identify it with

$$S_{G,M}(N) = \{p \in S_G(N) \mid p \text{ is finitely satisfiable in } M\}$$

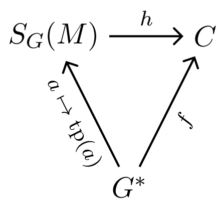
using the homeomorphisms induced by the identification of $\varphi(x,a)$ with $\varphi(M,a)$. They both have a natural G-flow structure given by translations.

Definition 4.22. Let C be a compact Hausdorff space and let[6] $f \colon G \to C$ is [externally] definable if for all closed disjoint C_1 and C_2 there are [externally] definable, disjount $D_1, D_2 \subseteq G$ such that $f^{-1}[C_i] \subseteq D_1$ for $i \in \{1,2\}$.

Note that it is "the same" definition that the one of "definable measure". This also agrees with this definition

Definition 4.23.
1. $f \colon G^* \to C$ is M-definable if for all closed $C_1 \subseteq C$ the set $f^{-1}[C_1]$ is type-definable.
2. $f \colon S_{G,M}(N)(\mathfrak{U}) \to C$ is externally definable if for all closed $C_1 \subseteq C$ the set $f^{-1}[C_1]$ is type-definable over N.

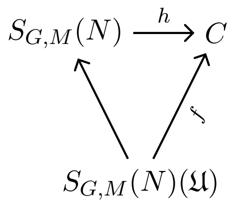
Remark 4.24.
1. $f \colon G^* \to C$ is M-definable if and only if there is a continuous $h \colon S_G(M) \to C$ such that



[5] Recall that an externally definable subset of M is something of the form $\varphi(M,a)$, where $a \in \mathfrak{U}$ may be outside a.

[6] This can be defined in the same way even for non-groups, but let's stay in our context-

2. $f\colon S_{G,M}(N)(\mathfrak{U}) \to C$ is externally definable if and only if there is a continuous $h\colon S_G(M) \to C$ such that



Fact 4.25. 1. If $f^*\colon G^* \to C$ is M-definable then $f^* \restriction G\colon G \to C$ is definable

2. If $f\colon G \to C$ is definable then there is a unique $f^*\colon G^* \to C$ extending f. This is given by $\{f^*(a)\} = \bigcap_{\varphi \in \mathrm{tp}(a/M)} \overline{f[\varphi(M)]}$.

Remark 4.26. There is a relation to continuous logic: you can think of it as a "C-valued continuous relation".

From now on, G is a discrete group.

Definition 4.27. A G-flow (G, X) is [externally] definable if $\forall x \in X$ the function $f_x\colon G \to X$ given by $g \mapsto g \cdot x$ is [externally] definable.

Fact 4.28. There is a unique (up to isomorphism) universal [externally] definable G-ambit.

The proof we saw before can be adapted to prove the previous fact, the only difficulty being showing that [externally] definable stuff is closed under products.

Fact 4.29. $(G, S_{G,M}(N), \mathrm{tp}(e/N))$ is the universal externally definable G-ambit.

Proof. Let us check that $(G, S_{G,M}(N))$ is externaly definable. Take any $p \in S_{G,M}(N)$ and $\varphi(x, n)$ satisfiable in M. Then

$$f_p(g) \in [\varphi] \Leftrightarrow g \cdot p \in [\varphi] \Leftrightarrow \varphi(x, n) \in gp \Leftrightarrow \models \varphi(g \cdot \alpha, n)$$

so $\{g \mid f_p(g) \in [\varphi]\}$ is externally definable by $\varphi(y \cdot \alpha, n)$.

We will not see the universal property. □

Fact 4.30. 1. If all types in $S_G(M)$ are definable, then $S_G(M) = S_{G,\mathrm{ext}}(M)$ and $(G, S_G(M), \mathrm{tp}(e/M))$ is the universal definable G-ambit.

2. It is not true in general that $(G, S_G(M))$ is the universal definable G-ambit. The problem is that it may not be the case that all types are definable (but it happens in $\mathbb{R}, \mathbb{Q}_p$; the fact that it holds in all models is equivalent to the stability of T).

3. In general[7], the universal definable G-ambit is of the form $(G, S_G(M)/E, \mathrm{tp}(e/M)/E)$, for some closed relation E. If the language is countable this is a Polish space and you can use descriptive set theory. This is *not* the case with $S_{G,\mathrm{ext}}$: it need not be Polish, even if the language is countable.

Corollary 4.31. There is a semigroup operation $*$ on $S_{G,M}(N)$, continuous on the left and extending the action of G (i.e. $\mathrm{tp}(g/N) * p = gp$). So $(S_{G,M}(N), *) \cong E(S_{G,M}(N))$, where the isomorphism is still ℓ.

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Remark 4.32. $p * q = \lim_i g_i q$ for any net $(g_i) \subseteq G$ such that $\lim g_i = p$.

There is also a more explicit characterization:

Fact 4.33. $p*q = \mathrm{tp}(a \cdot b/N)$ where $b \models q$ and a realizes the unique extension of p to a complete type over Nb (i.e. $N \cup \{b\}$) which is finitely satisfiable in M.

Why does this unique extension exists and is unique? Because also types over Nb finitely satisfiable in M correspond to elements of $S_{G,\mathrm{ext}}(M)$ and from this is easy to conclude.

Remark 4.34. If you take the full language on G you recover the classical discrete case and those type spaces are βG. Anyway the characterization is not that useful because the language is too complicated.

Proof of the Fact. it is enough to find $(g_i) \subseteq G$ such that $\lim_i \mathrm{tp}(g_i/N) = p$ and $\lim_i g_i \cdot q = \mathrm{tp}(a \cdot b/N)$. Take $\varphi(x) \in p$ and $\psi \in \mathrm{tp}(a \cdot b/N)$. Then by definition we have $\models \varphi(a) \wedge \psi(a \cdot b)$. Since $\mathrm{tp}(a/Nb)$ is finitely satisfiable in M, there is $g_{\varphi,\psi} \in G$ such that $\models \varphi(g_{\varphi,\psi}) \wedge \psi(g_{\varphi,\psi} \cdot b)$. Then[8] $\lim_{(\varphi,\psi)} \mathrm{tp}(g_{(\varphi,\psi)}/N) = p$. Then we have

$$\lim_{\varphi,\psi} g_{\varphi,\psi} \cdot q = \lim_{\varphi,\psi} \mathrm{tp}(g_{\varphi,\psi} \cdot b/N) = \mathrm{tp}(a \cdot b/N)$$

□

We now want to compare this new invariant (the Ellis group) with quotients by *connected components* a model theoretic invariant which we are now going to recall.

Definition 4.35. ${G^*_A}^{00}$ is the smallest A-type definable subgroup of G^* of bounded index, meaning that the index is bounded by the saturation of the monster (actually by $2^{|T|+|A|}$).

[7] Krupiński thinks that even without definability of types things can be worked around.
[8] The limit is a limit of nets.

Definition 4.36. ${G^*_A}^{000}$ (also called ${G^*_A}^{\infty}$) is the smallest A-invariant[9] subgroup of G^* of bounded index.

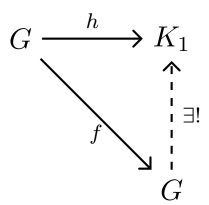
Fact 4.37. $G^*/{G^*_A}^{00}$ is normal, and the quotient is a compact Hausdorff topological group which is independent of the choice of $\mathfrak{U}$. Same thing for $G^*/{G^*_A}^{000}$, except that it is not necessarily Hausdorff. The topology is given by: $D \subseteq G^*/H$ is closed iff $\pi^{-1}[D]$ is type-definable (where $\pi\colon G^* \to G/H$).

Non-Hausdorffness of $G^*/{G^*_A}^{000}$ is a problem: while you interpret $G^*/{G^*_A}^{00}$ as a "classical mathematical object" (a topological group), how do you do that with $G^*/{G^*_A}^{000}$? Here topological dynamics will come handy.

This is a classical definition:

Definition 4.38 (Classical context: G a topological group)**.**

1. A compactification of G is a continuous homomorphism[10] $f\colon G \to K$ with dense image, where K is a compact Hausdorff group.
2. The Bohr compactification is a unique (up to isomorphism) universal compactification f of G, i.e. for all compactification $h\colon G \to K_1$ there is a unique factoring through f:



It can be constructed taking the product of all compactifications up to homomorphism (they are only a set (and not a proper class) because G must be dense in the compactification). Uniqueness follows by standard abstract nonsense.

Definition 4.39. Let G be definable in M

- An [externally] definable compactification of G is an [externally] definable homomorphism as above.
- The [externally] definable Bohr compactification of G is the universal [externally] definable compactification of G.

Fact 4.40. There exists a unique [externally] definable Bohr compactification of G.

[9] Under $\mathrm{Aut}(\mathfrak{U}/A)$.

[10] Not necessarily injective.

Proposition 4.41 (Gismatullin, Penazzi, Pillay)**.** $f\colon G \to G^*/{G_A^*}^{00}$ is the definable Bohr compactification of G.

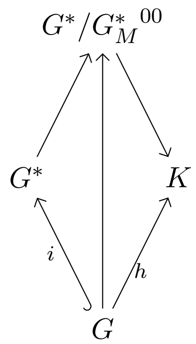
(no hypotheses on T, so this also applies to the classical case; anyway, again, then the language is too big so ${G_A^*}^{00}$ is kind of mysterious)

Example 4.42.

1. If $M = G = (\mathbb{Z}, +)$ then G^{*00} (which in this case does not depend on M) is $\bigcap_{0\neq n\in\mathbb{Z}} n\mathbb{Z}^*$ and $G^*/G^{*00} \cong \hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$ (basically, take a nonstandard integers and consider the remainders modulo n...)

2. $M = (\mathbb{R}, +, \cdot)$, $G = S^1$. Then G^{*00} is the group of infinitesimals. Then $G^*/G^{*00} = S^1(\mathbb{R})$.

Idea of the Proof of the Proposition. Let $h\colon G \to K$ be a definable compactification. Then $h \subseteq h^*G^* \to K$ which is M-definable and a homomorphism. Then $\mathrm{Ker}(h^*)$ is obviously the pre-image of a closed set, so it must be type-definable (over M) by definability, and it is of bounded index because it is bounded by $|K|$ (G is dense in it!). So $G^{*00}_M \subseteq \mathrm{Ker}(h^*)$. To show the other inclusion consider the diagram



then by uniqueness of the Bohr compactification the map $G^*/{G_M^*}^{00} \to K$ must be an homeomorphism (the rightmost triangle commutes!) □

From now on I is a minimal left ideal of $S_{G,M}(N)$, $u \in \mathrm{Idem}(I)$, so uI is the Ellis group of $S_{G,M}(N)$. Define $\hat{f}\colon S_{G,M}(N) \to G^*/{G_M^*}^{000}$ as $\mathrm{tp}(a/N) \mapsto a/{G_M^*}^{000}$. Is this well-defined? Yes.

Proposition 4.43. $\hat{f}$ is a well-defined, continuous semigroup epimorphism.

This was first done by Newelski for ${G_M^*}^{00}$, then Pillay extended it to ${G_M^*}^{000}$.

Proof. Well-definedness : if $\mathrm{tp}(a/N) = \mathrm{tp}(b/N)$ then $a \equiv_M b$ (in fact the same Lascar strong type, but since M is a model...). Having the same type over M is the finest bounded M-type-definable equivalence relation. Therefore $a/{G^*_M}^{000} = b/{G^*_M}^{000}$.

Continuity : Exercise

Hom If $p, q \in S_{G,M}(N)$ take $b \models q$ and $a \models p$ such that $\mathrm{tp}(a/Nb)$ is finitely satisfiable in M. Then $p * q = \mathrm{tp}(ab/N)$,

$$\hat{f}(p * q) = ab/{G^*_M}^{000} = a/{G^*_M}^{000} \cdot b/{G^*_M}^{000} = \hat{f}(p) \cdot \hat{f}(q)$$

Epi Take $a/{G^*_M}^{000}$. We have $\mathrm{tp}(a/M) \subseteq \mathrm{tp}(a^*/N) \in S_{G,M}(N)$. Then $\hat{f}(\mathrm{tp}(a^*/N)) = a/{G^*_M}^{000}$.

□

Corollary 4.44. $f = \hat{f} \restriction uI$ is still onto (hence is a group epimorphism).

Proof. By minimality $I = S_{G,M}(N) * u$. Therefore

$$\hat{f}[u * I] = \hat{f}[u * S_{G,M} * u] = \hat{f}(u) * \underbrace{\hat{f}[S_{G,M}(N)]} * \hat{f}(u)$$

□

notice that from here you recover the same result for $G^*/{G^*_M}^{00}$, since $/{G^*_M}^{000}$ is a subgroup of ${G^*_M}^{00}$, hence just take the projection between the quotient

Conjecture 4.45 (Newelski)**.** Under reasonable assumptions the map $\Theta\colon uI\colon G^*/{G^*_M}^{00}$ is an isomorphism.

(Krupiński and Gismatullin proved that in F_2 with full language $G^{00} \neq G^{000}$. Obviously there is no hope to have injectivity to $G^*/{G^*_M}^{000}$ when ${G^*_M}^{000} \neq /{G^*_M}^{00}$)

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Let us see some examples where the conjecture is true and where it is false:

Fact 4.46 (Newelski)**.** If $\mathrm{Th}(M)$ is stable then

1. $S_{G,\mathrm{ext}}(M) = S_G(M)$ (all types are definable) has a unique minimal left ideal and it is just the collection of all the generic types

$$\mathrm{Gen}(G) = \{p \in S_G(M) \mid \forall \varphi \in p\ \exists n\ \exists g_1, \dots, g_n G = \bigcup g_i \varphi\}$$

2. There is a unique idempotent in Gen(G), and it is exactly the unique generic type in G^{*00}. Moreover in this case $I = uI$ and $\Theta\colon uI \to G^*/G^{*00}$ is an isomorphism[11] of topological groups.

Proof. 1. Let $p \in S_G(M)$ be generic and consider $q \in I$ and $\varphi(x) \in p$. Then $\varphi(x)$ is generic, so there is $g \in G$ such that $\varphi(x) \in gq$. This means that $p \in \overline{Gq} = I$ because I is a minimal flow. Hence Gen(G) $\subseteq I$. Since one can check that Gen(G) is a flow, we have equality by minimality.

2. As each coset of G^{*00} contains a unique generic type (this is a theorem of stable group theory).

□

Fact 4.47 (Pillay)**.** More generally, "this" holds when G is *fsg* (e.g. definably compact groups in o-minimal theories)

.

Example 4.48. $M = (\mathbb{R}, +, \cdot)$ and $G = S^1 \subseteq \mathbb{R}^2$. There is still one minimal ideal, the one of generic types, but possibly different idempotents. Recall that the types here are either algebraic types (which are not generic) or cuts (either left p_a^- or right p_a^+). It is easy to see that

$$\{p_a^+ \mid a \in S_1\} \cup \{p_a^- \mid a \in S^1\}$$

and that the only idempotents are p_1^+ and p_1^-.

The Ellis groups are $p_1^+ I$ and $p_1^- I$, which are isomorphic to $S^1(\mathbb{R})$.

Theorem 4.49 (Chernikov, Simon)**.** More generally, if G is definably amenable[12] and T is NIP then Newelski's conjecture is true.

NIP is used to have Borel-definability of types and *Baire-generic compact domination.* Let us see some counterexamples:

Example 4.50. If $G^{*00} \neq G^{*000}$, as we already said, the conjecture is not true. Examples of this situation are

1. M is the two-sorted structure $((\mathbb{Z}, +), (\mathbb{R}, +, .))$, with no interaction between the two sorts and G is the universal cover of $\mathrm{SL}(2, \mathbb{R})$.

2. $M = (F_2, \text{full structure})$ and $G = F_2$.

Example 4.51 (Gismatullin, Penazzi, Pillay)**.** If $M = (\mathbb{R}, +, \cdot)$ and $G = \mathrm{SL}(2, \mathbb{R})$, then the conjecture is not true: it turns out that $|uI| = 2$, but $|G^*/G^{*000}| = 1$.

[11] We do not write ${G_M^*}^{00}$ because in stable (actually in NIP) it does not depend on the model.

[12] I.e. there is Keisler measure on G which is invariant by left translations.

Example 4.52 (Jayiella, Yao)**.** Analogous computations for groups definable in o-minimal expansions of RCF, which posses the compact-torsion-free decomposition.

Let's look at a deeper insight in Newelski's conjecture. This is work from Krupiński and Pillay.

There is the so-called τ-topology in uI, which is compact, T1, and makes $*$ separately continuous. Inversion is also continuous (but $*$ as a function of two arguments may not be τ-continuous). Consider

$$H(uI) = \bigcap_{U\ \tau\text{-nhbhd of } U} \operatorname{cl}_\tau /U$$

One can show that this is a normal subgroup of S_1 and

Fact 4.53. *$uM/H(uM)$ is a compact Hausdorff topological group.*

It is called the *generalized Bohr compactification*, but there is no time to explain this.

$$\begin{array}{ccc} uI & \xrightarrow{f} & G^*/G^*_M{}^{000} \\ & \searrow & \uparrow {\scriptstyle \exists!} \\ & & uI/H(uI) \end{array}$$

Theorem 4.54.

Where on $uI/H(uI)$ there is the quotient topology coming from the τ-topology.

Theorem 4.55. *$\bar{f}[\operatorname{cl}(\operatorname{Ker}(\bar{f}))] = G^*_M{}^{00}/G^*_M{}^{000} = \operatorname{cl}(\{e/G^*_M{}^{000}\})$ so $\bar{f}$ is a quotient topological mapping.*

Corollary 4.56. *$G^*/G^*_M{}^{000} = (uI/H(uI))/\operatorname{Ker}(\bar{f})$. (recall that it may be non-Hausdorff and the logic topology may even be trivial). Moreover $G^{*00}/G^*_M{}^{000} \operatorname{cl}(\operatorname{Ker}(\bar{f}))/\operatorname{Ker}(\bar{f})$.*

This has applications in Borel cardinalities and equivalence relations (did not write things down)

Corollary 4.57. *If f is an isomorphism then $\bar{f}$ is an isomorphism. Therefore $G^*_M{}^{00} = G^*_M{}^{000}$ (and the τ-topology is compact Hausdorff). Hence if the two connected components are distinct, even the strong form f Newelski's conjecture (with three zeros) is false.*

Proof. Points are closed in the τ topology. Apply the previous theorem. □

Theorem 4.58. *If all types in $S_G(M)$ are definable and G is definably strongly amenable (i.e. all definable proximal flows are trivial)*[13] *then $G^*_M{}^{00} = G^*_M{}^{000}$.*

[13] All nilpotent groups are strongly amenable, hence definably strongly amenable.

This was recently generalized to amenable groups by Krupiński and Pillay. In general it is an open problem to characterize the groups for which ${G^*_M}^{00} = {G^*_M}^{000}$.

This is one of the first papers by Newelski on the subject: L. Neweslki, *Topological dynamics of definable group actions*, JSL 74.1 (pp 50-72), 2009

There should be a paper by Newelski where he presents the Ellis semigroup as an inverse limits of definable groups (in stable context). Then a student of Hrushovski started studying stable semigroups.

The fact that a flow is equicontinuous is equivalent to the Ellis semigroup being a topological group.

Recent developments by Krupiński and Pillay are on arXiv (go to to the authors' webpages).

Example of compact-torsion-free decomposition: $\mathrm{SL}(2, \mathbb{R}) = H \cdot K$, where H is of the form $\begin{pmatrix} b & c \\ 0 & b^{-1} \end{pmatrix}$ with $b \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}$ and $K = \mathrm{SO}(2, \mathbb{R})$.