BPGMTC 2017

Probably Unreliable Notes

Rosario Mennuni

Contents

Readme

Disclaimer These notes have been typeset "on the fly" in LaTeX during the British Postgraduate Model Theory Conference 2017, held in Leeds from the 25th to the 27th of January 2017. They have *not* been reviewed, and were originally intended only for personal use. As a result, they may contain severe errors, huge omissions, and are in general only what I managed to write down. In particular, they are *not* official notes, and have not been reviewed by the speakers.

Because of the reasons above, these notes can be quite embarrassing, if not just pure nonsense. Please forgive me when this happens (it will), especially if this happens with notes from your talk.

Emails pointing out errors, mistakes, etc. are very welcome. In particular, if you spoke in the conference and want to correct these notes, or supplement them, or have them removed, please contact me.

Acknowledgements Thanks to Erick García-Ramirez for finding and correcting a lot of typos.

Notation Omissions are marked [like this], sometimes with some words saying what is being omitted, sometimes just with dots as in [...]. Also, if you see [slides] at the beginning of a talk, it means the talk had slides. These should be available on the conference website. Typically, talks with slides will have notes more incomprehensible/incomplete than usual. In general, I usually use square brackets for comments.

Info You can find these notes on `http://poisson.phc.unipi.it/~mennuni/Mennuni_BPGMTC17_Notes.pdf` (but they could be moved; in case, check my Leeds webpage[1]). You can contact me at `mmrm@leeds.ac.uk`. This version has been compiled on February 8, 2017. To get the source code click on the paper clip.

Rosario Mennuni

[1]Which does not exist yet, otherwise I would have linked that.

Chapter 1

Amador Martin-Pizarro – Mini-course on Equationality and Chain Conditions

1.1 25/01

The notion of equationality evolved from the work of Srour. It was then picked up by Pillay et al. Apparently nowadays there are not many people working in this area.

Notation 1.1. Fix a complete theory T in a language L with infinite models. Work inside a sufficiently saturated model $\mathfrak{U}$.

Definition 1.2. Consider x and y tuples of variables. A formula $\varphi(x, y)$ is *equational (over y)* iff the family of intersections of instances of $\varphi(x, b)$, for $b \in \mathfrak{U}$ has the DCC, i.e. there is no infinite sequence

$$\varphi(x, b_0) \supsetneq \varphi(x, b_0) \cap \varphi(x, b_1) \supsetneq \dots \bigcap_{i<n} \varphi(x, b_i) \supsetneq \dots$$

Example 1.3. Some examples:

1. $x = y$ is an equation: each instance only has one realisation.
2. If E is a definable equivalence relation, then $E(x, y)$ is an equation, more or less for the same reason: the classes of an equivalence relation are a partition.
3. If $\varphi(x, y) = \varphi(x)$ (i.e. it does not depend on y), then it is an equation: obviously, plugging parameters does not change the set.
4. If there is an underlying field on $\mathfrak{U}$, let $p(x, y) = \sum a_\alpha(y)x^\alpha$ be a multivariate polynomial[1]. The formula $\varphi(x, y) \coloneqq p(x, y) = 0$ is an

[1] α is a multi-index.

equation by Hilbert's Basis Theorem: just consider the ideal generated by the polynomials $p(x, b_i)$ and use the fact that it is finitely generated.

5. $x \neq y$ is *not* an equation. Note that, in particular, the notion of equationality is not preserved by negations[2].

6. If there is a definable linear order of the universe, then $x < y$ is *not* an equation, and neither is $x \geq y$.

7. An example of φ such that both φ and $\neg\varphi$ are equational is $E(x, y)$ in an equivalence relation with two classes.

Remark 1.4. If $\varphi(x, y)$ is an equation on y, then it is an equation on x (anyway, it is important that the partition is given: you cannot change it in general).

Proof. If not, there is a sequence $\{a_i\}_{i\in\mathbb{N}}$ such that $\varphi(a_0, y) \supsetneq \varphi(a_0, y) \cap \varphi(a_1, y) \supsetneq \ldots$. So, for each $n \in \mathbb{N}$, there is some $b_n \in \bigcap_{i<n} \varphi(a_i, y) \setminus \varphi(a_n, y)$. So, for each $n \in \mathbb{N}$, we have $\models \varphi(a_i, b_n)$ for $i < n$ and $\not\models \varphi(a_n, b_n)$. But this witnesses that

$$\varphi(x, b_n) \supsetneq \varphi(x, b_n) \cap \varphi(x, b_{n-1}) \supsetneq \ldots \supsetneq \bigcap_{i\leq n} \varphi(x, b_i)$$

By compactness, there is a sequence $\langle c_i \mid i \in \mathbb{N}\rangle$ such that $\varphi(x, c_0) \supsetneq \varphi(x, c_0) \cap \varphi(x, c_1) \supsetneq \ldots$. □

Corollary 1.5. $\varphi(x, y)$ is an equation if and only if there is no sequence $\langle a_i b_i \mid i < \mathbb{Z}\rangle$ such that $\models \varphi(a_i, b_j)$ for $i < j$ but $\not\models \varphi(a_j, b_j)$. In particular, equations do not have the order property, so "equational" implies "stable". Also, by usual arguments, we may assume that $\langle a_i b_i \mid i < \mathbb{Z}\rangle$ is indiscernible.

We say that negations may destroy equationality. What happens with other connectives?

Remark 1.6. If $\varphi_1(x, y), \varphi_2(x, y)$ are equations, so are $\varphi_1 \wedge \varphi_2$ and $\varphi_1 \vee \varphi_2$. In other words, equationality is preserved under *positive* boolean combinations.

Proof. For $\wedge$, note that if $(A \cap C) \cap (B \cap D) \subsetneq A \cap B$ then either $A \cap C \subsetneq A$ or $B \cap D \subsetneq B$. Use this and the pigeonhole principle. For the union is similar. □

What about quantifiers?

Remark 1.7. If $\varphi(x_1, x_2, y)$ is an equation, it need not be the case that $\exists x_1 \varphi$ is.

[2]As opposed to, say, having the order property, having the independence property,...

Example 1.8. Suppose there is a field around. We want to encode the formula $x_1 \neq y$, which we know is not equational. This can be done with $x_2(x_1 - y) = 1$, which is an equation, but $\exists x_2\, x_2(x_1 - y) = 1$ is not, since it is equivalent to $x_1 - y \neq 0$. Note that this is also an example of not being an equation if you change the partition of the variables.

Definition 1.9. A definable set $X_b = \varphi(x, b)$ is *Srour-closed* iff the family of finite intersections of conjugates, i.e. finite intersections of elements in the family $\{\varphi(x, b') \mid b' \equiv b\}$ has the DCC. So here we only allow parameters with the

Remark 1.10. If $\varphi(x, y)$ is an equation, then for all b the formula $\varphi(x, b)$ is Srour-closed.

Remark 1.11. If $X_b = \varphi(x, b)$ is Srour-closed, then there is a formula $\theta(y) \in \mathrm{tp}(b)$ such that $X_b = \varphi(x, b) \wedge \theta(b)$ and $\varphi(x, y) \wedge \theta(y)$ is an equation.

In particular, Srour-closed sets are exactly the definable set given by an instance of an equation (but not all formulas defining them are equations, in general).

Proof. Let $p = \mathrm{tp}(b)$. By compactness there is $n \in \mathbb{N}$ such that $p(y_1) \cup \ldots \cup p(y_n)$ says that there is no chain

$$\varphi(x, y_1) \supsetneq \varphi(x, y_1) \wedge \varphi(x, y_2) \supsetneq \ldots \supsetneq \bigcap_{i \leq n} \varphi(x, y_i)$$

Let $\theta(y) \in p$ be such that $\bigwedge \theta(y_i)$ already implies that there is no chain as above. Clearly, this θ does the job. $\square$

Definition 1.12. We say that T is *n-equational* (for $n \in \mathbb{N}$) iff every formula $\varphi(x, y)$ with $|x| = n$ is a boolean combination of equations $\psi(x, y)$. We say that T is *equational* iff it is n-equational for every $n \in \ltimes$.

Question 1.13 (Open)**.** Does T being 1-equational imply that T is equational?

Remark 1.14. If T is 1-equational, then it is stable.

Anyway, the other implication is not true. For a long time there was just one example by Hrushovski (even ω-stable; something called the free pseudo-plane colored with some colors), quite complicated (probably 1996 or 1997 or something like that). Sela has recently shown that the theory of the non-abelian free group[3] is stable but not equational. Proving that a theory is equational is usually difficult. Same thing to prove that stable theory is not equational. For proving that one is, usually the first step is

[3] All the free groups with $n \geq 2$ have the same theory; proved by Sela as well.

showing that every formula is a boolean combination of some "special" form, and then that these are equations (and there is no general technique for this last step).

One can try to prove that all strongly minimal theories are equational using Morley rank, but it is not clear that the rank should go down with intersections. In this aspect, Zariski dimension is stronger than Morley rank.

1.2 26/01

Remark 1.15. An obviously equivalent condition (modulo saturation) for a definable set X_b to be Srour-closed is for the family of finite intersection of its conjugates under automorphisms to have the descending chain condition.

Remark 1.16. In a stable theory types over models are definable.

Definition 1.17. A type $p \in S(M)$ is definable iff for all formulas $\varphi(x, y) \in L$ there is an $L(M)$-formula $\theta(y)$ such that for all $m \in M$ we have $\varphi(x, m) \in p \iff \models \theta(m)$.

Suppose that φ is an equation. Consider

$$\bigcap_{\substack{m \in M \\ \varphi(x,m) \in p}} \varphi(x, m)$$

By DCC, this is a finite intersection, hence an $L(M)$-definable set X. So in this case we can take as θ the $L(M)$-formula

$$\theta(y) := \forall x\ (x \in X \to \varphi(x, y))$$

It is clear that $\varphi(x, m) \in p$ implies that $X \subseteq \varphi(x, m)$, hence $m \models \theta$. Since p is complete, the other implication holds as well.

Even if equations are delicate (not preserved under many things), equationality is more robust:

Remark 1.18. These facts hold:

- Equationality is preserve under naming or forgetting parameters.
- Equationality is preserved under bi-interpretability. In particular, it passes to M^{eq}.

We will not see the proof of the above.

Isabel Müller mentioned not-1-ampleness, equivalent under strong minimality to 1-basedness. This is equivalent, in this context, to the following.

Definition 1.19. A definable set is *weakly normal* if for all $a \in X$ there are only finitely many distinct conjugates of X containing a.

Example 1.20. Equivalence classes are weakly normal.

Remark 1.21. If X is weakly normal, then it is Srour-closed.

Proof. Choose countably many distinct conjugates X_i of X_b. We must have $\bigcap_{i \in \omega X_i} = \emptyset$ by definition. By compactness, a finite intersection is already empty. □

Definition/Proposition 1.21.1. A stable theory[4] T is 1-based iff every definable set is a boolean combination of weakly normal definable sets.

Corollary 1.22. A stable 1-based theory is equational.

This is so far the more general result that the speaker knows saying when a theory is equational. 1-based is the first level of a certain hierarchy. The second level is not known:

Question 1.23 (Probably open)**.** Is every stable CM-trivial theory equational?

Fact 1.24. The above is true if the theory has finite Morley rank, the Morley rank coincides with the U-rank, and the U-rank is continuous. (Probably U-rank finite and continuous suffices).

The proof is kind of involved.

This definition is easier to work with than equations. It is an equivalent definition of equationality.

Definition 1.25. A set X is *indiscernibly closed (icl)* iff whenever $\langle a_i \mid i < \omega + 1\rangle$ is an $\emptyset$-indiscernible sequence such that $a_i \in X$ for all $i < \omega$, then $a_\omega \in X$.

This almost gives a closure operator, except it is not idempotent. It comes from the work of Junker and Lascar (The indiscernible topology: a mock Zariski topology).

Example 1.26. An example and a non-example:

- Let X be defined by $x = b$. If $a_1, a_2 \in X$, then $a_1 = a_2$, hence $a_\omega = a_1 \in X$ as well.
- Let X be defined by $x \neq b$, where $b \notin \mathrm{acl}(\emptyset)$. Take infinitely many different realisations of $\mathrm{tp}(b/\emptyset)$. Then we violate icl by just letting $a_\omega = b$.

Remark 1.27.

[4] Maybe one needs to work in T^{eq}.

1. If X is definable, we may assume that the indiscernible sequence is arbitrarily long and ordered as we want. So, for example, we may replace a_ω with a_0 and $i < \omega$ with $i > 0$ in the definition.

2. If X is definable over $\emptyset$, it is icl.

Lemma 1.28. A definable set X_b is Srour-closed if and only if it is icl.

Proof. ($\Rightarrow$) Otherwise, there is $\{a_i\}_{i\in\omega}$ such that $a_0 \notin X$ and $a_i \in X$ for $i > 0$. For any $n \in \omega$, choose an automorphism σ_n such that for all i we have $\sigma_n(a_i) = a_{n+1}$ (note that σ_0 is the identity). Now we have $X_b \supsetneq X_b \cap \sigma_1(X_b)$, since $a_1 \in X_b$ but $a_1 = \sigma_1(a_0) \notin \sigma_1(X_b)$. Similarly, $a_n \in \bigcap_{i<n} \sigma_i(X_b) \setminus \sigma_n(X_b)$, a contradiction.

($\Leftarrow$) If not, there are $\langle a_i b_i \mid i \in \mathbb{Z}\rangle$ indiscernible over the empty set, with $b_0 = b$, and φ an equation such that $X_b = \varphi(x, b)$ and $\models \varphi(a_i, b_j)$ if $i < j$ but $\not\models \varphi(a_i, b_i)$. Observe that $\langle a_i \mid i \leq 0\rangle$ is indiscernible, and that $\models \varphi(a_i, b_0)$ for $i > 0$ but $\not\models \varphi(a_0, b_0)$. □

We are now going to talk about *belles paires* of algebraically closed fields of characteristic q (q may be 0).

Definition 1.29. Let $E \prec K$ be a pair of algebraically closed fields of the same characteristic q. We say that (K, E) is a *belle paire* iff $\mathrm{tr}_{\deg} E = \mathrm{tr}_{\deg_E} K = \infty$.

Definition 1.30. A subfield $A \subseteq K$ is *tame*[5] iff $A \mathop{\smile\!\!\!\!\!|}^{\mathsf{ACF}}_{A\cap E} E$.

Remark 1.31. If $A \subseteq E$ then A is tame.

Exercise 1.32. Any two belles paires (K_1, E_1), (K_2, E_2) of the same characteristic are elementarily equivalent in the language L_P, i.e. the language of rings with a unary predicate for E.

Hint. This smells like back and forth. Consider the family of field isomorphisms $f\colon A \to B$, where $A \subseteq K_1$, $B \subseteq K_2$, A and B are fraction fields of finitely generated substructures, A tame in K_1, B tame in K_2, and $f(A \cap E_i) = B \cap E_2$. Show that it has the back-and-forth property and is not empty.

Note that we are not considering all partial isomorphisms, so this does not provide quantifier elimination (it does not have it). □

Corollary 1.33. The common theory T_P of belles paires of characteristic q is complete.

What should be axioms for T_P? Clearly, it has to contain

- ACF_q

[5] Non-standard terminology.

- $\exists x \notin P$
- P is rel. alg. closed: $\forall x_1, \ldots, x_n \, \forall y \, (y + \sum x_i y^i = 0 \land \bigwedge x_i \in P \to y \in P)$.

Exercise 1.34. An $\aleph_0$-saturated model of these axioms is a belle paire. Note that then, by Löwenheim-Skolem, these are axioms for belles paires.

Hint. E will have infinite transcendence degree and there should be an element π not in E. You can get another transcendental π^* by finding a sequence that converges in the logic topology, then iterate. □

1.3 27/01

Let T_P be the theory axiomatised with the axioms given right before Exercise 1.34.

Proposition 1.35. Work inside some sufficiently saturated $(K, E) \vDash T_P$, and let $A \subseteq K$ be tame, as defined yesterday. Then its L_P-type in T_P is determined by its L_P-quantifier-free-type. In particular, the theory T_P is ω-stable of Morley rank ω.

This theory could never be of finite Morley rank because there is a proper subfield: since the extension is proper, there are a lot of $a \in K$ transcendental over E; consider the polynomial rings $E[a]$. More precisely:

Anyway

Fact 1.36. If N is a small submodel,

- If $a \in E$ is transcendental over N, then $\mathrm{MR}(a/N) = 1$
- If $a \notin \widetilde{NE}$, then $\mathrm{MR}(a/N) = \omega$.

The goal now is to answer the following

Question 1.37. Is T_P equational?

Definition 1.38. An L_P-formula $\varphi(x)$ is *tame*[6] iff there are polynomials $p_1, \ldots, p_n \in \mathbb{Z}[x, z]$, homogeneous[7] in the variables[8] z, such that

$$\varphi(x) \equiv \exists \zeta \in P \left(\zeta \neq 0 \land \bigwedge_{i=1}^{n} p_i(x, \zeta) = 0 \right)$$

From now on, everything is joint work with Martin Ziegler.

[6] Again, non-standard terminology.
[7] Not necessarily of the same degree. We only need them to make sense in projective space.
[8] Everything is a tuple.

Proposition 1.39. In T_P every formula is a boolean combination of tame formulas.

In particular, if we want to show that a theory is equational, a good start could be to study the tame formulas.

Remark 1.40. Why homogeneous polynomials? Look at the Zariski closed set $xy - 1 = 0$. When we project it on x we get $x \neq 0$, which is constructible, but not closed. This does not happen with a *complete* variety, e.g. the projective ones; the definition is something like having projections (when you take the product with another variety) closed. If V is projective , its projection is Zariski closed (and we know that Zariski closed sets are equations).

Note that a tame formula contains anyway a quantifier $\exists \zeta \in P$. The point is trying to get rid of "$\in P$" to prove equationality.

Proposition 1.41. For any partition x, y of the variables, the tame formula $\varphi(x, y)$ is an equation.

Corollary 1.42. The theory T_P is equational.

Proof. By the previous Proposition and Proposition 1.39. □

Partial proof of the Proposition. Let $\varphi(x, y)$ be a tame formula. We show that for all $a \in K$ the formula $\varphi(x, a) \wedge x \in P$ is indiscernibly closed, i.e. that for all $L_P(\emptyset)$-indiscernible sequences $\langle b_i \mid i \leq \omega \rangle$ such that if for all $i < \omega$ we have $b_i \models \varphi(x, a) \wedge x \in P$, then $b_\omega \models \varphi(x, a) \wedge x \in P$.

That $\forall i < \omega \; b_i \in P \Rightarrow b_\omega \in P$ is clear by indiscernibility. Choose a basis c of $E(a)/E$. Write each monomial of a in terms of c, i.e. transform

$$\varphi(x, a) \equiv \exists \zeta \in P \; \left(\zeta \neq 0 \wedge \bigwedge p_j(x, a, \zeta) = 0\right)$$

into

$$\exists \zeta \in P \; \left(\zeta \neq 0 \wedge \bigwedge q_j(x, \zeta) \cdot c = 0\right)$$

where the $\cdot$ denotes some sort scalar product, notationally: first write p_i as $\sum q_{jk}(x, \zeta) \times m_{jk}(a)$, where $m_{jk}(a)$ is a monomial in a, and then rewrite a in terms of the base c. Now the $q_j(x, \zeta)$ are coefficients over E, homogeneous on ζ. Now

$$\models \varphi(b_i, a) \iff \exists \zeta \in P \; \left(\zeta \neq 0 \wedge \bigwedge q_j(b_i, \zeta) = 0\right)$$

because since we have written everything with coefficients over a basis, they all have to be zero. Since ACF eliminates quantifiers, $E \preceq K$ as fields. Then the formula above is equivalent to

$$\exists \zeta \; \left(\zeta \neq 0 \wedge \bigwedge q_j(b_i, \zeta) = 0\right)$$

This says that, for all $i < \omega$, b_i lies in the projection of a specific projective variety V, and projection of projective varieties are Zariski closed. We know that Zariski closed sets are indiscernibly closed, so $b_\omega \in V$.

This was one particular case (where the sequence lies in P). The general case can be reduced to this via some stability theory. $\square$

There is a different method using intersections which worked in characteristic 0. This is an attempt to consider a different theory, i.e. the one of separably closed fields. The ones of finite imperfection degree[9] are equational and stable. In infinite imperfection degree there are some problems. In that case the theory is stable and has quantifier elimination in a suitable language. The open question is

Question 1.43. Is the theory of separably closed fields of *infinite* imperfection degree equational?

A couple of words on some technical details.

In T_P, $a \equiv b$ if there are tame fields $A \ni a$, $B \ni b$ and an L_P-isomorphism $A \to B$ sending a to b. Let $V \subseteq E^n$ be a vector subspace. Take some basis $v_1, \ldots, v_k$ of V. We have

$$V = \{v \in E^n \mid v \wedge v_1 \wedge \ldots \wedge v_n = 0\}$$

where $v \wedge v_1 \wedge \ldots \wedge v_n \in \bigwedge^k E^n$ depends on the basis, but only changes by a scalar multiple. So the right object to associate to it is a point in projective space. Basically, we are considering Plücker coordinates $v \mapsto p_k(v)$ to embed the Grassmanian $\mathrm{Gr}_k(E^n)$ in a suitable projective space. Let $(x_1, \ldots, x_n) \to \{m_k(\bar{x})\}_{k \in \mathbb{N}}$ be a fixed enumeration of monomials on x. Then $a \equiv_{L_P} b$ if and only if for all k we have both

$$\dim_E \mathrm{Ann}(m_{i \leq k}(a)) = \dim_E \mathrm{Ann}(m_{i \leq k}(b))$$

and

$$p_k(\mathrm{Ann}_{i \leq k}(a)) \equiv^{\text{field}} p_k(\mathrm{Ann}_{i \leq k}(b))$$

This is linear algebra, and all of this can be made into determinants (which are homogeneous polynomials) being zero.

Question 1.44 (Open)**.** Which conditions allow to transfer equationality (say, preservation by taking belles paires)?

There is also a theory of *equational forking*.

[9]Which is related to having a certain subfield, and that is the relation with belles paires.

Chapter 2

Invited Talks

2.1 Boris Zilber – On Algebraically Closed Fields of Characteristic 1

[slides] Joint with J. A. Cruz Morales, work in progress.

Some model-theoretic technicalities of this are shared with work on Schanuel's conjecture.

It starts in algebraic geometry over $\mathbb{F}_1$. The idea would be, for example, to define a group over a field without being precise about the field. It is related to *non-additive geometry*: you look at fields, or rings, just with the multiplicative structure (but you somehow remember that they had an additive structure). It mimicks the classical scheme theory, but instead of the category of rings it uses the category of commutative monoids. Actually, there is no canonical setting (but there is a kind of common center). It extends ideas of *Arakelov geometry*: embeds number-theoretic schemes in a context where every characteristic is present. The idea is to have an algebraic geometry linking all characteristics. Model theoretically, these structures of different characteristic should not be orthogonal.

An $\mathbb{F}_1$-algebra is a monoid $\mathbb{U}$ with a "trace of addition" on it. For $\mathbb{F}_{1^n}$ take $\mathbb{U} = \mathbb{C}_n \cup \{0\}$, where $\mathbb{C}_n := \{\xi \in \mathbb{C} \mid \xi^n = 1\}$.

$\mathbb{Z}[\mathbb{U}] = \mathbb{Z}[\zeta_n]$ is the usual cyclotomic extension of degree n. $\mathfrak{p}$ prime ideal of $\mathbb{Z}[\mathbb{U}]$ containing p. $\mathbb{F}_1^{\mathrm{alg}} = \tilde{\mathbb{F}}_1$ corresponds to $\mathbb{U} = \mathbb{C}_\infty = \bigcup \mathbb{C}_n$. The "trace of addition": for each prime p, map $\mathrm{res}_\mathfrak{p}\colon \mathbb{Z}[\mathbb{U}] \to \tilde{\mathbb{F}}_p$, with $m_1u_1 + \ldots + m_nu_n \equiv 0 \pmod{p}$.

Equivalently, you can consider a multi-sorted structure $\tilde{\mathbb{F}}_1 = \langle \mathbb{U}, \tilde{\mathbb{F}}_2, \tilde{\mathbb{F}}_3, \ldots, \tilde{\mathbb{F}}_p, \ldots \rangle$, a "fusion" of algebraically closed fields $(\tilde{\mathbb{F}}_p, +, \cdot)$ with common multiplicative structure $(\mathbb{U}, \cdot)$ (the structure has the $\mathrm{res}_\mathfrak{p}$ maps; note that the structure depends on the choice of the residue maps).

Can one fuse distinct characteristic algebraically closed fields into one nice existentially closed structure? This is a challenge in terms of "fusion". The recipe for this comes from Hrushovski 1988-90 (Zilber was conjecturing at the time that it was not possible to do it (guess it's referring to the trichotomy).

Model theoretic questions: if this structure is really a generalisation of usual fields: what about things like Steinitz exchange?

In a saturated extension ${}^*\tilde{\mathbb{F}}_1$ of $\tilde{\mathbb{F}}_1$ define, where $\bar{u}(p) := \mathrm{res}_{\mathfrak{p}}(\bar{u})$ and $\mathrm{g.\,rk}$ is the group rank, the Hrushovski predimension

$$\delta(\bar{u}) = \min_{S \subseteq \mathrm{Pr}} \sum_{p \in S} \mathrm{tr}_{\deg} \mathbb{F}_p[\bar{u}(p)] - (|S| - 1) \cdot \mathrm{g.\,rk}\langle \bar{u} \rangle$$

The condition to get nice fusing is $\delta(\bar{u}) \geq 0$ for any tuple $\bar{u} \subseteq {}^*\mathbb{U}$.

This is a "Schanuel" conjecture SCF_1 for $\mathbb{F}_1$ geometry. Schanuel's conjecture presumes some transcendental function. What is transcendental here? The point is that the multi-sorted structure above is not very algebraic, it is more an analytic structure, given by the residue maps.

Supporting evidence for SCF_1: M. Bays noticed that SCF_1 implies this "Manin-Mumford" statement for $\tilde{\mathbb{F}}_\ell$, where ℓ is a prime different from p:

Let $W \subseteq \mathbb{G}_m^n(\tilde{\mathbb{F}}_\ell)$ and P the p-subgroup of $\mathbb{G}_m^n(\tilde{\mathbb{F}}_\ell)$. Then $W \cap P$ is contained in a finite number of cosets of a proper algebraic subgroup of $\mathbb{G}_m^n(\tilde{\mathbb{F}}_\ell)$.

Theorem 2.1 (Bogomolov[1])**.** The "Manin-Mumford" statement for $\tilde{\mathbb{F}}_\ell$ does hold.

This can be extended to the case of P to be the $\{p_1, \ldots, p_k\}$-subgroup.

Lemma 2.2. SCF_1 implies the Conjecture on Intersections with Tori over $\mathbb{F}_1$, i.e.

Conjecture 2.3. For any collection W_S of subvarieties $W_p \subseteq \mathbb{G}_m^n(\tilde{\mathbb{F}}_p)$, $p \in S$, such that $\sum_{p \in S} \dim W_p < (|S|-1)n$ there is a finite collection of proper tori $\tau_{W,S}$ in $\mathbb{G}_m^n(\tilde{\mathbb{F}}_0)$ satisfying $\forall \bar{u} \ \bigwedge_{p \in S} \bar{u}(p) \in W_p \Rightarrow \bigvee_{T \in \tau_{W,S}} \bar{u} \in T$.

SCF_1 can be reformulated as a first-order axiom scheme [missed it].

This is a $\mathbb{F}_1$-nullstellensatz[2] (algebraic closedness axioms): Let W_p, $p \in S$ be a collection of algebraic subvarieties of $\mathbb{G}_m^n(\tilde{\mathbb{F}}_p)$. Axiom AC_1: suppose $\bigwedge_{p \in S} \bar{u}(p) \in W_p$ does not contradict SCF_1, then such an $u \in \mathbb{U}^n$ does exist. Formal statement: there is a finite family $\mathcal{M}_W$ of $(n \times m)$-matrices such that

$$\bigwedge_{M \in \mathcal{M}_W} \sum_{p \in S} \dim W_p^M(\zeta) \geq (|S| - 1)\,\mathrm{rk}(M) \Rightarrow \exists u \in \mathbb{U}^n \bigwedge_{p \in S} u(p) \in W_p(\zeta)$$

where ζ are parameters (varieties now depend on parameters), taken from $\mathbb{U}$. So, the statement is uniform in parameters.

[1]Zilber asked him, he replied the next morning with a half-page argument that took a while to read.

[2]The terminology comes from Macintyre.

What are Zariski-closed sets and algebraic closure here? [no time, anyway]: Here there is no direct analogue of Chevalley's theorem: the projection of a closed set is not necessarily constructible (no elimination of quantifiers).

Axioms for ACF_1:

$$\mathsf{ACF}_1(\mathbb{U}) \coloneqq \mathsf{SCF}_1 \cup \mathsf{AC}_1 \cup \mathrm{Dgm}(\mathbb{U})$$

where the last disjunct is the diagram of $\mathbb{U}$, i.e. the collection of all algebraic relations satisfied by the $u(p)$ as u varies in $\mathbb{U}$ and p in the primes (depends on res_p). For finite $x, Y \subseteq {}^*\mathbb{U}$ we can define $d(x/Y) \coloneqq \min\{\delta(xYz) - \delta(Y) \mid z \subseteq {}^*\mathbb{U}\}$ and $\mathrm{cl}(Y) \coloneqq \{x \in {}^*\mathbb{Y} \mid d(x/Y) \leq 0\}$ (one still has to prove that d is actually a dimension).

Say that a model has CCP, the *countable closure property*, if whenever Y is finite, $\mathrm{cl}(Y)$ is countable.

Theorem 2.4. Assume that the axioms of $\mathsf{ACF}_1(\mathbb{U})$ are consistent[3]. Then for each such $\mathbb{U}$ the theory $\mathrm{ACF}_1(\mathbb{U})$ is complete, ω-stable with $\mathrm{MR}(\mathbb{U}) = \omega$ and near model complete (everything is a boolean combination of existentially [positive?] definable sets. Moreover, for every cardinal κ; there is a unique model satisfying CCP of transcendence degree κ. The operator cl in every model satisfies the exchange principle.

Note the analogy with the Steinitz theory on ACF. (the cl there is not acl: it is the *analytic closure*: in a pseudo-analytic closure, you consider an analytic variety of dimension 0, i.e. you take the zero-set of. . .)

Further tests for the axioms: let p_1, p_2, p_3 be distinct primes, $a_{i1}, a_{i2} \in \mathbb{F}_{p_i}$ all non-zero, and consider the system of equations $a_{i1}u_1(p_i) + a_{i2}u_2(p_i) = 1$, seeking for $\bar{u} \in \mathbb{U}^2$ such that $\bar{u}(p_i)$ satisfies the equalities. Prove that the system of 3 equations has at most finitely many solutions (axioms SCF_1). Prove that the system of 2 equations always has infinitely many solutions (axioms ACF_1).

2.2 Zoé Chatzidakis – Difference Fields and Applications to Algebraic Dynamics

We will not see much of algebraic dynamics.

Definition 2.5. A *difference field* is a field with a distinguished endomorphism σ.

Since we are in a field, σ will always be injective. It will not be surjective in general.

Example 2.6.

[3]Recall it depends on the residue maps.

- For $K, \tilde{K}$, we can take $\sigma \in \mathrm{Aut}(\tilde{K}/K)$ and consider $(\tilde{K}, \sigma)$. Here the orbit of every element is finite.
- In $\mathbb{C}(t)$, take $\sigma \restriction \mathbb{C} = \mathrm{id}$ and $\sigma(t) = t + 1$.
- $\tilde{\mathbb{F}}_p$ (or indeed the algebraic closure of any field of characteristic p) and, for $q = p^n$, the map $x \mapsto x^q$. This is in general not onto unless K is perfect. For instance, if $K = \widetilde{\mathbb{F}_p(t)}$ then t is not a qth power.

In the 1960s, Ax studied the theory T_f of all finite fields and its infinite models, i.e. the pseudofinite fields. They are elementarily equivalent to ultraproducts of finite fields.

Every absolutely irreducible variety over a pseudo-finite field will have a point (indeed infinitely many) as a corollary of [Lang-Weil?]

What about $\mathrm{Th}(\tilde{\mathbb{F}}_p, x \mapsto x^q)$? Does it have a model companion? I.e. is the class of existentially closed models elementary?

(note that σ in this case is definable; but not uniformly in q, so...)

Fact 2.7. The existentially closed difference fields form an elementary class. Their theory is called ACFA.

What does it mean for a difference field to be existentially closed? The typical term is some $t(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n, \sigma(x_n), \ldots, \sigma^m(x_1), \ldots]$ (difference polynomial/system polynomial). Existentially closed means that each difference equation that has a solution in an elementary extension has a solution in K.

The main names for the algebra of these structures are Ritt and Cohn. Difference fields behave similarly to differential fields. Names in their model theory: Macintyre, van den Dries, Wood. Then Chatzidakis, Hrushovski, and then Peterzil, Pillay, ..., Medvedev.

What are the main facts in ACFA?

- σ is onto (because you can make it onto in an extension)
- $\mathrm{Fix}(\sigma) = \{a \in K \mid \sigma(a) = a\}$ is a subfield, and it turns out to be pseudofinite. In some sense, ACFA is to the theory of pseudofinite fields what differentially closed fields is to the theory of algebraically closed ones.
- It is (by definition) model-complete.
- It is decidable; this comes from a
- Partial quantifier elimination result, in the sense of the Fact below.

Fact 2.8. Let (E, σ) be an algebraically closed difference field, $\sigma(E) = E$. Look at $L(E)$. Then $\mathrm{qfDiag}(E) \cup \mathsf{ACFA}$ is complete.

Start with $\bar{a} \in \mathfrak{U}$. Let A be the field generated by all $\sigma^i(\bar{a})$, for $i \in \mathbb{Z}$. Then look at $\tilde{A}, \sigma$.

Example 2.9. Fix(σ) is not algebraically closed. If char(K) $\neq 2$, consider the formula

$$\exists y \ \big(\sigma(y) = y \wedge \forall y \ z^2 = y \to \sigma(z) = -z\big)$$

This is not equivalent to a quantifier-free formula.

Fact 2.10. ACFA has elimination of imaginaries.

So, if $\varphi(\bar{x}, \bar{b}) \subseteq \mathfrak{U}^n$, and you define $E_\varphi(\bar{y}, \bar{z}) := \forall x \ \varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{z})$, this is a $\emptyset$-definable equivalence relation. By elimination of imaginaries, there is $\bar{c} \in \mathfrak{U}^n$ such that for all $\tau \in \mathrm{Aut}(\mathfrak{U}, \sigma)$ (obviously, τ must commute with σ), then $\tau(X) = X$ if and only if τ fixes the elements of $\bar{c}$.

In this respect is better behaved that the theory of pseudofinite fields, where you need to add constants to have elimination of imaginaries.

Fact 2.11. ACFA is supersimple of rank ω.

In particular, there is a good notion of independence. What does it mean here to have $A = \mathrm{acl}(A)$? If this happens, then A must be a field, we must have $A = \sigma(A)$, and $A = \tilde{A}$ (i.e. A must be algebraically closed as a field).

Fact 2.12. It turns out this is enough to characterise algebraic closure in ACFA.

The independence relation satisfies

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff \mathrm{acl}(CA) \mathop{\smile\!\!\!\!\!\!|}^{\mathsf{ACF}}_{\mathrm{acl}(C)} \mathrm{acl}(BC)$$

where $\mathop{\smile\!\!\!\!\!\!|}^{\mathsf{ACF}}$ indicates independence in the ACF sense (the acl are in the ACFA sense).

There is a notion of rank: $\mathrm{SU}(\bar{a}/E = \mathrm{acl}(E)) = 0$ iff $\bar{a} \in E$, and $\mathrm{SU}(\bar{a}/E = \mathrm{acl}(E)) = 1$ iff if $\bar{a} \notin E$, $F \supseteq E$, and $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_E F$, then $\bar{a} \in \mathrm{acl}(F)$.

Example 2.13.

- $\sigma(x) = f(x)$, $f \in E[x]$, $f(a) = \sigma(a)$, $\mathrm{acl}(Ea) = \widetilde{K(a)}$
- $\sigma^2(x) = x^2 + 1$
- $\tilde{\sigma}(x) = x^2 + 1$ (in characteristic $\neq 2$). Chatzidakis and Hrushovski showed that this is strongly minimal.

[quick definition of 1-based]

Example 2.14. Fields are not 1-based: consider $ac + bd = 1$, where a, b, c are independent. $\widetilde{\mathbb{Q}(a,b)} \cap \widetilde{\mathbb{Q}(c,d)} = \widetilde{\mathbb{Q}}$.

Let p be a type over E of SU-rank 1. This gives a pregeometry (P, acl). This can be:

- Trivial: $\mathrm{acl}(A) \cap P = \bigcup_{a \in A} \mathrm{acl}(a) \cap P$
- Group-like 1-based; related to the existence of a definable group
- Field-like. In characteristic 0 this is related to $\mathrm{Fix}(\sigma)$; in characteristic ℓ also $\sigma^m(x) = x^{\ell^n}$.

Example 2.15. $\sigma(x) = x^2 + 1$ is trivial in characteristic $\neq 2$. $\sigma(x) = x^2$ is group-like 1-based, and $\sigma(x) = x$ is field-like.

In characteristic 0 there are additional properties: in some sense, the only instability comes from $\mathrm{Fix}(\sigma)$.

An application of this was to prove bounds for Manin-Mumford (Hrushovski).

Take a semi-abelian variety G defined over $\widetilde{\mathbb{Q}}$ (think of $(\mathfrak{U} \setminus \{'\}, \times)^n$). Let X be a subvariety. Then $X \cap \mathrm{Tor}(G) = \bigcup_{i=1}^{N} a_i + \mathrm{Tor}(G_i)$, where G_i is a subgroup of G.

Strategy of proof: find $\sigma \in \mathrm{Aut}(\widetilde{\mathbb{Q}}/k)$ and $f \in \mathbb{Z}[X]$ such that all points of $\mathrm{Tor}(G)$ satisfy $f(\sigma)(x) = 0$ and for all n we have $(f(X), X^n - 1) = 1$.

Take $f(x) = x^2 + 3x + 1$ and $B = \sigma^2(x) + 3\sigma(x) + x = 0$ (where $+$ is the group law of G).

What is algebraic dynamics? Take an irreducible algebraic variety V (affine, projective,…) defined over $K = \tilde{K}$ and take a rational map $V \to V$, also defined over K.

Example 2.16. $V = \mathbb{A}^1(\mathfrak{U}) = \mathfrak{U}$ and $\varphi(x) = \frac{x+1}{x-2}$

Let $I(V)$ be associated prime ideal in $K[y_1, \dots, y_n]$. You can think of $K[\bar{y}]/I(V)$ as a generic point of the variety (if you embed it in $\mathfrak{U}$). You can take the fraction field of this ring and call it $K(V)$. Since you can compose them with φ, you can consider $(K(V), \varphi^*)$, where $\varphi^*(f(\bar{y})) = f(\varphi(\bar{y}))$. The image of φ^* is a subfield of $K(V)$, and this is a difference field.

In algebraic dynamics there is a notion of *preperiodic point*: a point p such that there are $m \neq n$ such that $\varphi^{(m)}(p) = \varphi^{(n)}(p)$. This is for points, but you can also define it for varieties (setwise or, if you prefer, for the code of the variety).

Conjecture 2.17 (Zhange)**.** A variety is preperiodic iff the set of its preperiodic points is Zariski dense.

The conjecture turned out to be false. Scanlon and Medvedev looked at $\mathfrak{U}^n \to \mathfrak{U}^n$ for $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$, for $f_i \in \mathfrak{U}[x_i]$ they completely characterised what the preperiodic varieties are.

2.3 Jaroslav Nešetřil – Structural Limits and Dichotomy of Sparse Classes

[slides]

Old dream: all asymptotic properties from single "limit structure". Examples: Fraïssé (direct) limit, fractals (e.g. scaling), structural limits. We deal with the latter.

For fractals there is a correspondence between planar quadrangulations and rooted trees.

Example 2.18 (Left limits)**.** Dense limits: $G_1, G_2, \ldots$ sequence of graphs converging if and only if forall finite graphs F

$$t(F, G_n) = \frac{|\hom(F, G_n)|}{|G_n|^{|F|}}$$

converges. In other words is the probability that a random map $F \to G_n$ is an homomorphism.

Works for dense graphs.

Example 2.19. There is an homomorphism from G to the triangle if and only if the chromatic number of G is ≤ 3.

[motivation and history (from statistic, computer science,...)] For sparse graphs, say of bounded degree:

Example 2.20 (Local limits)**.** $G_1, G_2, \ldots$ converging if and only if for all connected F with a distinguished point r

$$\frac{|\{v \mid B_p(G_n, v) \cong (F, r)\}|}{|G_n|}$$

converging.

[motivation/history] Related to Cayley graphs, group theory,...

Unifying approach: structural limits: $G_1, \ldots$ sequence of finite structures X-converging (X a fragment of first-order logic) iff for all $\varphi \in X$ the sequence $\langle \varphi, G_n \rangle$ is converging, where the *Stone bracket* is

$$\begin{aligned}\langle \varphi, G_n \rangle &= \mathrm{Prob}(G_n \vDash \varphi) \\ &= \frac{|\{(v_1, \ldots, v_k) \mid G_n \vDash \varphi(v_1, \ldots, v_k)\}|}{|G_n|^k} \\ &= \varphi(G_n)/|G_n|^k\end{aligned}$$

Example 2.21. $\varphi(x_1, x_2) = \exists x \;\; E(x, x_1) landE(x, x_2) \wedge x_1 \neq x_2$, saying there is a path of length 2 between x_1 and x_2. Then $\varphi(G)$ are the pair of vertices for which there is such a path, and $\langle \varphi, G \rangle$ is the probability that two vertices are joined by a path of length 2.

If X is the fragment of quantifier-free formulas you get left convergence. Local formulas give local convergence, sentences give elementary convergence. But you can of course consider all the FO sentences.

"Magical observation" (Ph. Kolaitis): there is $f\colon F \to G$ if and only if $G \vDash \bigwedge_{i,j \in E(F)} E(x_i, x_j)$ (for some assignment of the x_i; put an $\exists$ in front).

By cleaning you can prove that (G_n) is left-convergent iff (G_n) is quantifier-free convergent [and similar results] If (G_n) converges elementarily to an ultrahomogeneous graph then FO-convergence of (G_n) is equivalent to QF-convergent and to L-convergence.

Now that we have these notions of convergence, what are the limits? We have three types of limit objects: distributional (probabilistic), non-standard (logic), analytic. Examples for left limits (dense): respectively the exchangeable random graph, ultraproduct + Loeb measure, graphon [examples for other contexts]. For structural limits in general: invariant distribution (Nesetril+Pom '12), ultraproduct+Loeb measure (Nesetril+Pom '12), modeling (sometimes) (Nesetril+Pom '16).

Invariant distribution: recall Stone duality between a Boolean subalgebra X of $\mathsf{FO}(\mathcal{L})$ and $S(X)$.

Theorem 2.22. For every X-convergent sequence $(G_n)_{n \in\ mbN}$ there is a unique probability measure μ on $S(X)$ such that for every formula $\varphi(x_1, \ldots, x_p) \in H$ the following holds:

$$\int_{S(X)} \chi_{[\varphi]}(x)\, \mathrm{d}\mu(x) = \lim_{n \to \infty} \langle \varphi, G_n \rangle$$

Theorem 2.23 (Nesetril, Pom '12)**.** There are maps $G \mapsto \mu_G$ and $\varphi \mapsto f_\varphi$ such that

- $G \mapsto \mu_G$ is injective
- $\langle \varphi, G \rangle = \int_{S(X)} f_\varphi \, \mathrm{d}\mu_G$

...

[...]

Theorem 2.24. Let (G_n) be a sequence of finite L-structure, $\mathcal{U}$ a non-principal ultrafilter. Then there is a measure ν on the ultraproduct $\tilde{G} = \prod_{\mathcal{U}} G_n$ such that for every first order formula $\varphi(x_1, \ldots, x_k)$ holds:

$$\int \cdots \int_{\tilde{G}^k} [\ldots] = [\ldots]$$

This works for every language. Can you get a separable model?

Extreme case: $\mathbb{G}$ (or M) standard Borel space such that every first order definable set is measurable. [...] Say that G_n converges to $M(X, \mu)$ iff for every $\varphi(x_1, \ldots, x_k)$

$$\int [ldots] \to [\ldots]$$

[Gaifman locality]

Theorem 2.25 (Nesetril-Pom '14)**.** Assume that every FO-convergent sequence (G_n) of elements of $\mathcal{C}$ converges to a modeling. Then $\mathcal{C}$ is nowhere dense class of graphs.

[sparse-dense dichotomy in terms of nowhere dense]

Definition 2.26. $\mathcal{C}$ is somewhere dense if there is p such that every finite graph G has a subdivision[4] $\tilde{G} \in \mathcal{C}$ which uses at most p (subdivision) vertices per every edge.

Nowhere dense is equivalent to: for all p there is $N(p)$ such that there are no $K_{n(p)}^{(p)}$. This is related to a lot of things.

[...]

Theorem 2.27 (Nesetril, Pom', 16+)**.** A monotone class of graphs $\mathcal{C}$ is nowhere dense if and only if every convergent sequence of graphs in $\mathcal{C}$ converges to a modeling.

[proof sketch]

[4]Putting degree 2 vertices in edges.

Chapter 3

Contributed Talks – 25/01/17

3.1 Isabel Müller – Zilber's Trichotomy and Counterexamples

3.1.1 Historical Overview

It all started with Morley's categoricity theory. Turns out the building blocks of κ-categorical theories are strongly minimal sets.

Fact 3.1. If T is strongly minimal, then acl has the exchange property, i.e. if $a \in \operatorname{acl}(bC) \setminus \operatorname{acl}(C)$, then $b \in \operatorname{acl}(aC)$.

It follows that the acl operator defines a *pregeometry*. Zilber proposed a conjecture about these.

Zilber's Trichotomy Conjecture

Conjecture 3.2. If T is a strongly minimal theory, then the pregeometry defined by acl falls in one of these cases

1. "set-like", disintegrated, i.e. $\operatorname{acl}(B) = \bigcup_{b \in B} \operatorname{acl}(b)$
2. "vector space-like", locally modular, i.e. see later
3. "field-like", i.e. there is a field interpretable.

There is a natural notion of dimension in all of the three cases above: namely

1. Cardinality
2. Linear dimension
3. Transcendence degree

Actually, dimension functions and geometries are in bijective correspondence. All those functions are *submodular*, i.e. $\dim(AB) \leq \dim(A) + \dim(B) - \dim(A \cap B)$.

Definition 3.3. A geometry is *modular* iff the above is an equality, i.e. $\dim(AB) \leq \dim(A) + \dim(B) - \dim(A \cap B)$, it is *locally modular* if this happens whenever $\dim(A \cap B) > 0$.

The conjecture was proven false by Hrushovski. This gave rise to

1. Weakenings of the conjecture (i.e. restricting it to certain classes structures)
2. Study of Hrushovski's method

Hrushovski's Counterexample

The idea is to start with a dimension function that violates the conjecture and use a Fraïssé limit to construct a geometry which has that function as its dimension.

The main ingredients are:

- Let $L = \{R\}$, where R is a ternary predicate.
- Let $\mathcal{C}^0$ be the class of L-structures where R is symmetric and irreflexive.
- Set $\delta(A) = |A| - |R(A)|$, which is in $\mathbb{Z}$ for A finite.
- For $A \subseteq_{\text{fin}} X \in \mathcal{C}^0$, say $A \leq X$ is *strong* iff $\delta(B/A) = \delta(AB) - \delta(A) \geq 0$ for all $B \subseteq_{\text{fin}} X$.
- Let $\mathcal{C}^1$ be the class of structures in $\mathcal{C}^0$ with non-negative δ, i.e. $\{X \in \mathcal{C}^0 \mid \emptyset \leq X\}$.
- The intersection of strong sets is again strong, whence for all $A \subseteq_{\text{fin}} X$ there is some set that we call $\text{cl}(A)$ such that $A \subseteq \text{cl}(A) \leq X$, and $\text{cl}(A)$ is minimal such. Note that since $\leq$ only looks at finite things, $\text{cl}(A)$ will be finite if A is. Set $d(A) = \delta(\text{cl}(A))$.
- Choose $\mathcal{C}_M \subseteq \mathcal{C}^1$ such that in the limit M_μ satisfies that $a \in \text{acl}(B)$ if and only if $d(a/B) = 0$. The idea is to throw away multiple copies of the same structure (this is what that μ does).
- M_μ is strongly minimal (it is easy to see that there is a unique non-algebraic type by looking at quantifier-free types and using properties of Fraïssé limits)
- M_μ is not in one of the first two cases of the construction [missed why]
- To see that there is no field one shows that M_μ is "CM-trivial".

Ampleness

No time to give the definition but: for every $n \in \mathbb{N}$ a theory "can be" n-ample. T is 1-ample if and only if it is not 1-based (locally modular) and it is 2-ample if and only if it is not CM-trivial. Algebraically closed fields are n-ample for all n. Ampleness is a hierarchy: $n-1$-ample implies n-ample. The question was: how far is the conjecture to be true? Can we find counterexamples to all levels of the hierarchy? It was first proved that the hierarchy is strict: there is, for all n, a stable structure which is n-ample but not $n+1$-ample. It remained open if there is a finite rank counterexample. Müller and Tent answered this.

3.1.2 Incidence Geometries

There is some geometery of type $\bullet \overset{n}{\text{---}} \bullet \overset{n}{\text{---}} \bullet$ which is almost strongly minimal, 2-ample, not 3-ample.

3.1.3 Definition

A generalised n-gon is a bipartite graph of diameter n and girth $2n$. [example of 3-gon and points-and-lines interpretation]

Explanation of the diagram: each point stands for a sort (points, lines, planes), so you have a tripartite graph. The n says that if you look at the bipartite graph there it is an n-gon.

An example of $\bullet \overset{\infty}{\text{---}} \bullet$ is the free pseudoplane, which is ω-stable, of infinite rank, 1-ample, not 2-ample.

Tent built geometries of type $\bullet \overset{n}{\text{---}} \bullet$ which as almost strongly minimal, 1-ample, not 2-ample.

Another thing constructed by Pillay and [?], the free pseudo-space, is of type $\bullet \overset{\infty}{\text{---}} \bullet \overset{\infty}{\text{---}} \bullet$ ω-stable of infinite rank, 2-ample and 3-ample.

The first examples of k-ample, not $k+1$-ample are geometries of type $\bullet \overset{\infty}{\text{---}} \bullet \overset{\infty}{\text{---}} \bullet \quad \cdots \quad \bullet \overset{\infty}{\text{---}} \bullet$

Another think to look at is stuff of type $\bullet \overset{n}{\text{---}} \bullet \overset{n}{\text{---}} \bullet \quad \cdots \quad \bullet \overset{n}{\text{---}} \bullet$ There are some technicalities involved in dealing with the Hrushovski constructions.

Question: why all the n are the same? Answer: it is just easier since you can use duality arguments.

3.2 Daoud Siniora – Hall's Universal Group and Ample Generics

[slides] Done under supervision of D. MacPherson.

Let L be a countable first-order language. Let M be a countably infinite L-structure M. $\mathrm{Aut}(M)$ is a Polish group, since it is closed in $\mathrm{Sym}(\mathbb{N})$. Does it have ample generics? We will focus in this talk on homogenous M, in particular for Philip Hall's universal locally finite group.

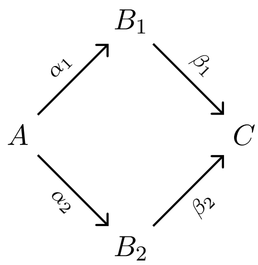
Definition 3.4. A countable structure M is *homogenous* iff every isomorphism between finitely generated substructures of M extends to an automorphism of M.

Remark 3.5. If M is locally finite, "finitely generated" is the same as "finite".

Definition 3.6. The *age* of M is the calss of all finitely generated L-structures which can be embedded in M.

Definition 3.7. Let $\mathcal{C}$ be a class of finitely generated L-structures. We call it an *amalgamation class* iff it is countable up to iso and has the hereditary property, the joint embedding property, and the amalgamation property.

The amalgamation property says



Theorem 3.8 (Fraïssé 1954)**.** If $\mathcal{C}$ is an amalgamation class, there is a unique up toiso homogenous L-structure M, which we call the limit of $\mathcal{C}$, such that $\mathcal{C}$ is the age of M.

Proposition 3.9 (Neumann)**.** The calss of all finite groups has the amalgamation property.

Proof. Let A, B, C be finite groups such that $A \leq B$ and $A \leq C$. Let $S = \{s_1, \dots, s_n\}$ be a set of representatives for right cosets of A in B. $[\dots]$ We ca write $B = AS$ and $C = AT$. The amalgam is $G = \mathrm{Sym}(A \times S \times T)$, with embeddings $\varphi\colon B \to G$ defined as $\varphi(b) = \varphi_b$, defined as $\varphi_b(a, s, t) = ([bas]^A, [bas]^S, t)$ (where $[-]^A$ is the A-coordinate), and $\psi(C \to G$ defined as $\psi(c) = \psi(c)$, where $\psi_c(a, s, t) = ([cat]^A, s, [cat]^T)$. □

The above is basically a generalisation of Cayley's theorem.

Theorem 3.10 (Hall)**.** There exists a unique countably infinite locally finite group $\mathbb{H}$ such that:

- Every finite group can be embedded in $\mathbb{H}$, and
- Any isomorphism between finite subgroups of $\mathbb{H}$ extends to an inner[1] automorphism of $\mathbb{H}$.

In other words, $\mathbb{H}$ is the Fraïssé limit of the class of finite groups.

Let $G = \mathrm{Aut}(M) \leq \mathrm{Sym}(\mathbb{N})$. Endow G with the pointwise convergence toplogy, generated by basic open sets of the form

$$G(\bar{a}, \bar{b}) = \{g \in G \mid g(\bar{a}) = \bar{b}\}$$

where $\bar{a}, \bar{b} \in \mathbb{N}^n$ and $N \in \omega$ (note that this not empty by homogeneity, which tells us we can extend g). These are the cosets of pointwise stabilisers of finite tuples, and it makes G into a Polish group. Recall that

Theorem 3.11 (Baire category Theorem)**.** Suppose X is a complete metric space. Then every countable intersection of dense open subsets of X is dense in X.

Definition 3.12. $C \subseteq X$ is *comeagre* if it contains a countable intersection of dense open sets.

Fact 3.13. The class of comeagre sets forms a δ-filter, i.e. a filter closed under countable intersections.

Let M be a countably infinite structure and put $G = \mathrm{Aut}(M)$. Consider the action of G by diagonal conjugation $g{\cdot}(h_1, \ldots, h_n) = (gh_1g^{-1}, \ldots, gh_ng^{-1})$.

Definition 3.14. M has n-generic automorphisms if G has a comeagre orbit on G^n in its action by diagonal conjugation. We say t has ample generics iff it has n-generics for all n.

When $n = 1$ it means that G has a comeagre conjugacy class. Note that same class means that automorphisms act "the same": they have the same cycle type, i.e. are the same modulo relabeling the domain.

Example 3.15 (Examples by Truss, Hodkinson, HHLS, Slecky)**.**

Suppose $\mathcal{C}$ is an amalgamation class of finite structures.

Definition 3.16. Let $n \in \omega$; an n-system over $\mathcal{C}$ is a tuple $S = \langle A, p_1, \ldots, p_n \rangle$, where $A \in \mathcal{C}$ and each p_i is a partial automorphism of A.

Let $\mathcal{C}^n$ be the calss of all n-systems. There is a natural notion of embeddings between systems: e.g. we say that $\langle A, p \rangle \to \langle B, f \rangle$ if $A \subseteq B$ and $p \subseteq f$.

[1] I.e. given by conjugation.

Theorem 3.17 (Kechris-Rosendal)**.** Suppose M is the Fraïssé limit of $\mathcal{C}$. Then $\mathrm{Aut}(M)$ has ample generics iff [...].

We will adopt the technique introduced by HHLS. It involves two properties, EPPA and APA, which imply the weak amalgamation property for $\mathcal{C}^n$ for all n [hence ample generics, by the theorem above]

Definition 3.18. A class $\mathcal{C}$ of finite L-structures has the Extension Property for Partial Automorphisms iff for every $A \in \mathcal{C}$ there is $B \in \mathcal{C}$ containing A as a substructure such that every partial automorphism of A extends to an automorphism of B.

Theorem 3.19 (Hrushovski)**.** The class of all finite graphs has EPPA.

Theorem 3.20 (Herwig)**.** The class of all finite K_n-free graphs has EPPA.

[more examples]

Theorem 3.21 (Hodkinson-Otto)**.** Let L be a finite relational language. Then the class of all finite L-structure has the Gaifman-clique faithful EPPA.

Theorem 3.22 (Siniora-Solecki)**.** Any free amalgamation class over a finite relational language has coherent[2] EPPA.

Theorem 3.23 (Hall)**.** The class of all finite groups has EPPA.

Definition 3.24. We say that $\mathcal{C}$ has the Amalgamation Property with Automorphisms iff whenever $A, B_1, B_2 \in \mathcal{C}$ with $A \subseteq B_1, B_2$ there is an amalgam $C \in \mathcal{C}$ such that whenever $f \in \mathrm{Aut}(B_1)$ and $g \in \mathrm{Aut}(B_2)$ such that $\forall a \in A\ f(a) = g(a)$ then there is $h \in \mathrm{Aut}(C)$ extending $f \cup g$.

Lemma 3.25. The class of finite groups has the amalgamation property with automorphisms.

Proof Ingredients.

1. The amalgamation property for finite groups
2. Free product of groups with amalgamation and its universal property (it is the pushout in the category of groups).

□

So the class of finite groups has both EPPA and APA, hence

Theorem 3.26 (Siniora)**.** Philip Hall's group has ample generics.

Song has proven the same result recently, independently.

[2] I.e. preserves composition.

Definition 3.27. A Polish group has the small index property if every subgroup of small index ($< \mathfrak{c}$) is open.

(the other implication is always true: open subgroups have small index).

Theorem 3.28 (HHLS and KR)**.** Suppose that M has ample generics. Then $\mathrm{Aut}(M)$ has the small index property.

Another consequence of ample generics:

Definition 3.29. If G isnot finitely generated, the cofinality of G is the smallest κ such that G can be written as a union of a chain of length κ of proper subgroups.

Theorem 3.30 (HHLS and KR)**.** If M is ω-categorical with ample generics, then $\mathrm{Aut}(M)$ has uncountable cofinality.

Another one:

Definition 3.31. Let G be a group which is not finitely generated. It has the Bergman property iff whenever $E \subseteq G$ and $1 \in E = E^{-1}$ and $G = \langle E \rangle$ there is $k \in \omega$ such that $G = E^k$.

Theorem 3.32. If M is ω-categorical and have ample generics then $\mathrm{Aut}(M)$ has ample generics.

3.3 Milette Tseelon-Riis – The Generalised Shift Graph

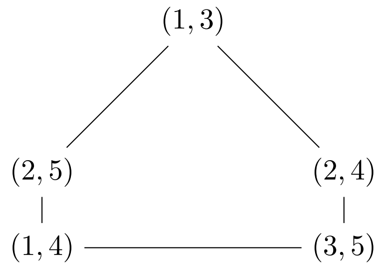
[I missed the beginning of this talk; since the speaker previously gave another talk on this same topic I am merging the notes I have for that one; this means the contents of these notes may differ from what was actually in the talk]

Start with a totally ordered set $(S, <)$ and a "code". Vertices in our graph will be n-tuples of S. The "code" tells us what n is and when two vertices are neighbours.

Example 3.33. Let $S = \{1, 2, 3, 4, 5\}$ with the natural order and $n = 2$. Connect[3] (a, b) and (c, d) if $a < c < b < d$ (or the symmetric thing).

How does this graph look like? (not drawing the nodes who are disconnected from everything)

[3]We assume that tuples are ordered, i.e. for every (a, b) we have $a < b$.



Let now $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ and suppose they are connected.. Say that for example

$$x_1 < y_1 < y_2 < x_2 = y3 < x_3 < \ldots$$

Color x_1 and x_3 with color 1, y_1 and y_2 with color 2 and $x_2 = y_3$ with color 3 (depending on in which color they are). Call *code* or *type* the sequence of colors (in the example, $1, 2, 2, 3, 1, \ldots$). From that, you can figure out who n is and where you have edges. Call it τ and represent the graph as (G, τ).

Fact 3.34. A code will be a sequence of 1, 2 and 3s with an equal number of 1 and 2s. From it you can reconstruct the edge relation (and n).

Fact 3.35. n is the number of 1s plus the number of 3s.

Example 3.36. What does $S = 5$ and $\tau = 12$ give? Here we will have $n = 1$ and the graph is going to be complete.

Example 3.37. $G(\omega + 1, 12) = G(\omega, 12)) = G(\mathbb{Z}, 12)$ is the complete graph on $\aleph_0$ vertices.

Example 3.38. If $\tau = 123$ and $S = 5$ then there are 4 connected components, isomorphic to K_1, K_2, K_3 and K_4. Analogously $G = (\omega, 123)$ is the disconnected sum of all the K_n for $n < \omega$. On the other hand $G = (\omega{+}1, 123)$ will also contain K_ω, and $G(\mathbb{Z}, 123)$ will be ω copies of K_ω. Notice that $G(\omega + \omega, 123) \cong G(\omega + \omega + 1, 123)$.

When are $G(\omega, 123)$ and $G(\beta, 123)$ isomorphic, for β an ordinal? Answer: for β of the form $\omega + \omega + \alpha$, where $\alpha < \omega_1$.

Example 3.39. $\tau = 132$. Here $(a, b)E(b, c)$. Note that a graph here will have an isolated point iff S has two endpoints.

Question 3.40. Are $G(\omega, 132)$ and $G(\omega + \omega, 132)$ isomorphic? No, the second one has a copy of $K_{\omega,\omega}$ in the vertices (n, ω) and $(\omega, \omega + n)$. The first one does not: any attempt with some (a, n) and (n, b) will have the "left" partition finite.

Question 3.41. Are there some α and β such that $G(\alpha, 132) \cong G(\beta, 132)$?

The answer is no.

Fact 3.42. $G(\omega, 132) \not\cong G(\omega+1, 132)$ and $G(\omega, 132) \not\cong G(\omega+\omega, 132)$.

Proof. In $G(\omega+1, 132)$ there is an isolated point: $(0, \omega)$, and there is no such in $G(\omega, 132)$. What about $G(\omega+\omega, 132)$? It contains the complete bipartite graph $K_{\omega,\omega}$, while $G(\omega, 132)$ does not. □

And actually

Theorem 3.43. If $\alpha \neq \beta$ are different ordinals, $G(\alpha, 132) \not\cong G(\beta, 132)$.

Proof. If α or β are finite, just count the number of vertices. Otherwise, look at a graph of the form $G(\alpha, 132)$ we will define a set $\bar{0}$ which will correspond (a posteriori) to the vertices starting with 0; similarly, we will define $\bar{\lambda}$ for each ordinal λ, and in the end we will have that α is the least ordinal such that $V(G) = \bigcap_{\lambda<\alpha} \bar{\lambda}$. Let $\bar{0}$ be given by

- it is a maximal co-clique, and
- pairwise disjoint neighbour sets

We claim that $\bar{0} = \{(0, \beta) \mid \beta > 0\}$. Suppose $\not\subseteq$. Then $(x, y) \in \bar{0}$ with $x \neq 0$. Then $(0, y) \in \bar{0}$ because $\bar{0}$ is a maximal co-clique, but (y, z) lies in both neighbour sets. By maximality $\bar{0}$ must be equal to $\{(0, \beta) \mid \beta > 0\}$.

In general, define $\bar{\lambda} \subseteq V(G)$ as

- maximal co-clique in $V(G) \setminus \bigcup_{\gamma<\lambda} \bar{\gamma}$, and
- pairwise disjoint neighbour sets.

We claim that, again, $\bar{\lambda} = \{(\lambda, \beta) \mid \beta > \lambda\}$. If this was not the case, then there is some vertex $(x, y) \in \bar{\lambda}$ with $x \notin \lambda$, so $x > \lambda$ (we removed all points before that). This implies as above that $(\lambda, y) \in \bar{\lambda}$, and again we find two meeting neighbour sets. □

[Isabel points out there may be an analogy with Cantor-Bendixson; the proof above sounds like removing isolated points]

The same proof works for $1333\ldots3332$. Anyway, the same statement is true for $11111\ldots333333\ldots2222\ldots2$.

3.4 Sebastian Eterovic – A Schanuel Property for the j Function

The j function is a very important function in number theory. It is a holomorphic, surjective function $\mathbb{H} \to \mathbb{C}$ ($\mathbb{H}$ is the upper half plane).

It satisfies some algebraic relations and some differential equations. Recall that $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $\mathbb{H}$ (Moebius transformations $gx = \frac{ax+b}{cx+d}$). If $g \in \mathbb{H}$

and $g \in \mathrm{GL}_2^+(\mathbb{Q})$, if $z = gz$ then we have $\Phi_N(g(z), j(g(z)) = 0$, where Φ_N is a certain polynomial in $\mathbb{Z}[x, y]$, and $\Phi_N(j(x), j(y)) = 0$. Here $N = \det(N(g)g)$.

On the differential side, we have a differential equation

$$0 = \frac{j'''}{j'} - \frac{3}{2}\left(\frac{j''}{j'}\right)^2 + \frac{j^2 + Aj + B}{j^2(j-1728)^2(j')^2}$$

where $A, B \in \mathbb{Z}$ (just to show that it is with rational coeffients, etc).

3.4.1 j-fields

The idea is the have a model theory for j. We call these structures j-fields. We have a field K, a distinguished $D \subseteq K$ which has to be a $\mathrm{GL}_2(\mathbb{Q})$-set (plays the role of the upper-half plane). Notice that we are not including the $^+$: we do not want to introduce an order to say what is positive. If we want to extend j to $\mathbb{H}^-$ we can just use Schwarz reflection $j(z) := \overline{j(\bar{z})}$. We impose these constraints on $j, j', j'', j''' \colon D \to K$ (no derivatives for now; think of j^{stuff} just as symbols):

- algebraic relations (not first-order; they are $\mathcal{L}_{\omega_1,\omega}$)
- differential equation
- a technical condition:

$$j(z) = 0 \lor j(z) = 1728 \lor j'(z) = 0 \to \bigvee_{g \in \mathrm{GL}_2(\mathbb{Q})} gz = z$$

  i.e. the troublesome points for the differential equation are only the "mandatory" ones.

The things above define $\mathbb{Q}$. [Zilber: could you define $\mathbb{R}$ there? Answer: you can take the complement of D, but you cannot prove it is a real closed field] Despite this, something can be done.

The purpose is to prove a transcendence statement for j. The strategy is: follow ideas for a similar statement for exponentiation (Bay, Kirby, Wilkie, 2010). Why should this work? The BKW result relies on the Ax-Schanuel Theorem, and in 2015 a version of this theorem for the j function appeared.

3.4.2 j-derivations

A j-derivation is a derivation $\partial \colon K \to K$ satisfying

- $\partial(j(z)) = j'(z)\partial z$
- $\partial(j'(z)) = j''(z)\partial z$
- $\partial(j''(z)) = j'''(z)\partial z$

We will see that there are non-trivial j-derivations on $\mathbb{C}$.

Notation 3.44. Given $A \subseteq K$ and $a \in K$, we say that $a \in \mathrm{jcl}(A)$ iff for every j-derivation such that $\partial(A) = 0$ we have $\partial(a) = 0$.

Under the axioms above, this is a pregeometry. We call the associated dimension $\dim^j$ (j-dimension). If $a \notin \mathrm{jcl}(\emptyset)$ then a is, in some sense, very generic.

3.4.3 Non-trivial j-derivations on $\mathbb{C}$

We say that relations $z = gz$ can be seen as algebraic relations $\Phi_N(j(z), j(gz) = 0)$ for a suitable N. For $g \in \mathrm{SL}_2(\mathbb{Z})$ the polynomial Φ_N happens to be $x - y$, so the condition is $j(z) = j(gz)$.

[fundamental domain $\mathcal{F}$]

Consider the expansion $\overline{\mathbb{R}}$ of the real field given by

$$\overline{\mathbb{R}} := \langle \mathbb{R}, \mathrm{Re}(j), \mathrm{im}(j), \mathrm{Re}(j'), \mathrm{im}(j'), \mathrm{Re}(j''), \mathrm{im}(j'') \rangle$$

where all those functions are restricted to $\mathcal{F}$. Peterzil-Starchenko proved that $\overline{\mathbb{R}}$ is o-minimal.

Let $\tau \in \overline{\mathbb{R}}$ be such that $\tau \notin \mathrm{dcl}(\emptyset)$ (there is one since $\mathrm{dcl}(\emptyset)$ is countable). It happens that

$$\forall a \in \mathbb{R}\ \exists f_a \colon \mathbb{R} \to \mathbb{R}\ f_a(\tau) = a$$

(e.g. the constant function). The point is that any $\emptyset$-*definable* function doing this needs to be differentiable around a [probably we should allow f_a to be $\{a\}$-definable because there not enough $\emptyset$-definable functions; but we want specifically the function to be non-constant otherwise the derivation below becomes trivial]. We can define a derivation $\partial \colon \mathbb{R} \to \mathbb{R}$ as $a \mapsto \frac{\mathrm{d}f_a}{\mathrm{d}x}(\tau)$. Even if the function is not unique, all of them have the same derivative in τ, so this is well-defined; also, it is a derivation and respects the real and imaginary parts of $j, j', j'' \restriction \mathcal{F}$.

If $a = \tau$ then you can take the identity and you get a non-trivial derivation with $\partial(\tau) = 1$.

Now define a derivation $\partial \colon \mathbb{C} \to \mathbb{C}$ as $a + ib \mapsto \partial(a) + i\partial(b)$. It can be shown that ∂ respects j, j', j'' (the full functions, not just on $\mathcal{F}$).

Given $A \subseteq K$, define $\mathrm{gcl}(A)$ to be the union of the $\mathrm{GL}_2(\mathbb{Q})$-orbits of points in A. This turns out to be a pregeometry, hence it has an associated dimension $\dim^g(A \mid B)$, namely the number of orbits of points in A which are not produced by points in B.

Theorem 3.45 (Ax-Schanuel Theorem for j, Pila-Tsimermann 2015)**.** For j-fields, if $\bar{z} \in D$ and $C \subseteq K$ is jcl-closed, then $\mathrm{tr}_{\deg}(\bar{z}, j(\bar{z}), j'(\bar{z}), j''(\bar{z}))(C) \geq 3\dim^g(\bar{z}(C)) + \dim^j(\bar{z}/C)$.

Theorem 3.46. Let $\tau_1, \ldots, \tau \in D$ be jcl-independent. Let $g_i \in \ operatornameGL_2(\mathbb{Q}(\tau_i))$, assume $g_i D \subseteq D$ and are not a multiple of someone in $\mathrm{GL}_2(\mathbb{Q})$. Let $\bar{z} \in D$. Suppose any of the following:

(a) $\bar{z}, \bar{g}\bar{z}$ are gcl-independent, or

(b) $\bar{z} \in C$ and gcl-independent but $gz = g$, $g \in \mathrm{GL}_2(\mathbb{Q})$.

Then

$$\mathrm{tr}_{\deg}\left(j(\bar{z}), j'(\bar{z}), j''(\bar{z}), j(\bar{g}\bar{z}), j'(\bar{g}\bar{z}), j''(\bar{g}\bar{z})\right) \geq 3nm$$

3.5 Alexander Jones – A Semantic Approach to Pathological Satisfaction Classes

[slides]

We will look at the language of arithmetic and as theory PA. The standard model of PA is $\mathbb{N}$. We will denote by M the non-standard models.

Countable nonstandard models can be used to prove incompleteness results and are also used for reverse mathematics. Their structure is isomorphic to $\mathbb{N} \cup (\mathbb{Z} \times \mathbb{Q})$, i.e. the standard natural numbers followed by $\mathbb{Z}$-chains that are denselyu ordered as per the rationals (all models, not just the saturated ones).

Expand the language L_A to $L_A \cup \{S\}$, where S is a two-place satisfaction relation. A satisfaction calss is a set of Gödel codes of sentences that are compositional linked, it respects the logical connection between sentences.

S is closed under $\wedge, \neg$. If $\varphi(i) \in S$ for all $i \in M$, then $\forall x\ \varphi(i) \in S$. They have a natural axiomatisation given by the theory CT^-.

The proper definition of a satisfaction class is more technical [axioms]. They meet Tarksi's material adequacy condition as definitions of truth: $M \models \varphi$ if and only if $(M, S) \models S(\ulcorner\varphi\urcorner, c)$. In other words, they are quite boring structures over the standard model of arithmetic. Over a nonstandard model, there are "nonstandard sentences".

Theorem 3.47 (Lachlan's)**.** If M admits a satisfaction class S, then M is recursively saturated[4].

Theorem 3.48 (KKL)**.** If M is countable and recursively saturated, then it has a satisfaction class.

If M is countable and has a satisfaction class, then it has $2^{\aleph_0}$ such satisfaction classes.

Many nonstandard sentences are intuitively false, but can be considered true by a satisfaction class, e.g.

[4] Quite a technical definition; it means that if you have a recursive type which is finitely satisfied...

1. $\delta_a^{0\neq 0}$ for a nonstandard a, where $\delta_0^{0\neq 0}$ is $0 \neq 0$ and $\delta_{n+1}^{0\neq 0}$ is $\delta_n^{0\neq 0} \vee \delta_n^{0\neq 0}$ for all $n \in M$.

2. For a nonstandard a, the formula $\exists x_0, x_1, \ldots, x_a[0 \neq 0]$

3. For a nonstandard a, the formula $\exists x_0 \forall x_1 \exists x_2, \ldots, \exists 2a[\varphi] \leftrightarrow \neg$

These can be contained in a satisfaction class because they can be consistently introduced in any "new" $\mathbb{Z}$-chain of M. Whilst a satisfaction calss is locally inductive around a$\mathbb{Z}$-chain, the composition clauses cannot stretch beyond this, as our met-induction does not stretch beyond ω. Thismeans one pathology in a satisfaction class actually results in a whole controversial family. It also menas that intuitively true sentences come out as false. This looks like a failure of Tarski's intended material adequacy: false sentences can be contained in a satisfacton class. Can we improve $\models$ to judge these as false? Can we then adapt the theory of satisfaction classes accordingly?

Robinson defined a notion of $\models^*$ for simple finite-rank nonstandard formulas which works.

Definition 3.49. Define these two notion by simultaneous induction.

A simple formula is one which is a standard formula or is a conjunction of (potentially nonstandard-many) simple formulas, disjunction of pnm simple formulas, negation of a simple formula of pnm quantifiers over a simple formula.

Define a rank function over simple formulas as a standard formula receiving rank $'$. A sequence of conjunctions, disjunctions, quantifications (maybe alternating) or a negation symbol of a formula of rank n increases the rank by 1.

Robinson then extends the standard notion of $M \models \varphi$ to one of $M \models^* \varphi$, where φ can be nonstandard (simple and of (standard-)finite rank). The only tricky part is the quantifiers: if $\varphi = Q\psi$, where Q is a pnm block of quantifiers, take the Skolem-normal form of φ and then remove the universal quantifiers for ξ. $M \models^* \varphi$ iff $M \models^* \xi(\ldots)$ for all possible substitutions in ξ. Here, for example, the Skolem normal form of $\forall x \exists y\ \varphi(x, y)$ is $\forall x\ (\varphi(x, f(x)))$ for a Skolem function f (PA is strong enough to have definable Skolem functions (it has minima of definable sets by induction)).

With this definition we have that $M \not\models^* \varphi$ where φ is a pathology.

Example 3.50. Consider $\exists x_0 \forall x_1 \exists x_2, \ldots, \exists 2a \delta_0^{(0\neq 0)}$. This is a pathology since it can be true in a satisfaction class, but is made false by $\models^*$. We get that $M \models^* (0 = 1)$. [more stuff]

Let $\mathrm{CT}^- + \models^*$ denote the class of models of arithmetic with satisfaction classes closd under this new material adequacy condition (Robinson semantics).

Theorem 3.51 (Jones)**.** The theory of $\mathrm{CT}^- + \vDash^*$ is given by $\mathrm{CT}^- + \mathrm{I}\Delta_0$.

Corollary 3.52 (Cieśliński)**.** The theory of $\mathrm{CT}^- + \vDash^*$ is equivalent to:

- $\mathrm{CT}^- + \forall\varphi\forall c[\mathrm{Prov}_{\mathrm{FOL}}(\varphi) \to S(\ulcorner\varphi\urcorner, c)]$
- $\mathrm{CT}^- + \forall\varphi\forall c[\mathrm{Prov}^S_{\mathsf{PA}}(\varphi) \to S(\ulcorner\varphi\urcorner, c)]$
- $\mathrm{CT}^- + \mathrm{Con}_S$

Models of CT^- are well-known, and so are models of CT^- with full induction. The model theory of $\mathrm{CT}^- + \mathrm{I}\Delta_0$ is not known yet.

Chapter 4

Contributed Talks – 26/01/17

4.1 Christian D'Elbée – Minimal (unstable) Expansions of $(\mathbb{Z}, +, 0)$

4.1.1 Expansions of $(\mathbb{Z}, +, 0)$

Definition 4.1. Take two structures M and N with the same domain. We say that N is an *expansion* of M (or M is a *reduct* of N) iff every definable set in M (possibly with parameters) is also definable in N. We say that M and N are *interdefinable* iff each is a reduct of the other one.

Note that by definition adding or removing parameters does not change anything.

Example 4.2. $(\mathbb{R}, <)$ is a reduct of $(\mathbb{R}, +, \cdot)$.

In terms of classification, we will talk of stable, simple, NIP, and dp-minimal. The latter is to NIP what strongly minimal is to stable.

Fact 4.3. $(\mathbb{Z}, +, 0)$ is stable of rank 1 and eliminates quantifiers in the language $\{+, -, 0, 1, (P_n)_n\}$, where

$$P_n(x) \equiv \exists y \ \underbrace{y + \ldots + y}_{n \text{ times}} = x$$

Note that expanding with a finite set does nothing, expanding with multiplication destroys tameness.

The first stable expansion of $(\mathbb{Z}, +, 0)$ was found in 2014, namely $(\mathbb{Z}, +, 0, \pi_q)$, where $\pi_q = \{q^n \mid n \in \mathbb{N}, q \geq 2\}$. Another expansion is [?].

An well-known unstable expansion is Presburger arithmetic, i.e. $\mathrm{Th}(\mathbb{Z}, +, 0, <)$.

Theorem 4.4 (Alouf-d'Elbée)**.** Consider $(\mathbb{Z}, +, 0, |_p)$, where $|_p$ is the divisibility predicate associated to the p-adic valuation. This is dp-minimal unstable.

Theorem 4.5. $(\mathbb{Z}, +, 0, |_{p_1}, \dots, |_{p_n})$ is of dp-rank n.

Question 4.6 (Goodrick)**.** Characterise the $A \subseteq \mathbb{Z}^n$ such that $(\mathbb{Z}, +, 0, A)$ is stable.

If A is the set of prime, a certain conjecture in number theory (Dixon's conjecture) implies that it is supersimple unstability. Some fresh news are:

Theorem 4.7 (Conant, 2017)**.** If $A \subseteq \mathbb{Z}$ has some "good sparsity" assumption, then $(\mathbb{Z}, +, 0, A)$ is stable.

4.1.2 Minimal expansions

Definition 4.8. An expansion N of M is *minimal* iff every N' which are simultaneously expansions of M and reducts of N is interdefinable with either N or M.

Example 4.9 (Marker, 1990)**.** Expanding $(\mathbb{C}, +, \cdot)$ with $(\mathbb{C}, +, \cdot, \mathbb{R})$ is minimal.

Theorem 4.10 (Conant, 2016)**.** $(\mathbb{Z}, +, 0, <)$ is a minimal expansion of $(\mathbb{Z}, +, 0)$.

Proof Idea. This is similar to the idea in Marker. If $A \subseteq \mathbb{Z}^n$ is definable in $(\mathbb{Z}, +, 0, <)$ but not in $(\mathbb{Z}, +, 0)$, then $(\mathbb{Z}, +, 0, A)$ defines the order. □

Theorem 4.11 (Alouf-d'Elbée)**.** $(\mathbb{Z}, +, 0, |_p)$ is a minimal expansion of $(\mathbb{Z}, +, 0)$.

The method of proof for this one also gives a proof for the Conant result.

Proof Idea. Let $(\mathbb{Z}, +, 0, \dots)$ be an unstable reduct of $(\mathbb{Z}, +, 0, <)$ in some language L. In particular, it is dp-minimal because it is a reduct of a dp-minimal structure. Now $(\mathbb{Z}, +, 0, \dots)$ is unstable and $(\mathbb{Z}, +, 0)$ is stable. Since stability is already detected in dimension one, there is a model $M \equiv (\mathbb{Z}, +, 0, <)$ and an L-formula $\varphi(x, y)$ with $|x| = 1$, and some $b \in M$ such that $\varphi(M, b)$ is *not* $\{+, 0\}$-definable. E.g. iff $\varphi(M, b) = [0, a]$ for some infinite non-standard a then $\mathbb{N}$ is externally definable in $(\mathbb{Z}, +, 0, \dots)$, since you can write it as $\varphi(M, b) \cap \mathbb{Z}$. Actually, it can be shown that it is definable (this fits in the general idea of honest definitions for externally definable sets in NIP theories). The strategy is saying that if $x \in A$ then $x + 1 \in A$, and that if $x \in A$ then $x - 1 \in A$, except for two points (who will be 0 and a).

The general case can be deduced from the case above. □

Remark 4.12. The proof above does not need the domain to be $\mathbb{Z}$. We could have started wit any non-standard model of Presburger, i.e. if $(N, +, 0, \dots)$ is unstable, where $(N, +, 0, <) \equiv (\mathbb{Z}, +, 0, <)$, then $<$ is definable in $\{+, 0, \dots\}$.

Remark 4.13. The remark above is in some sense optimal: it is not true for a stable reduct.

In the $|_p$ case the idea is analogous, except you deal with swiss cheeses.

4.1.3 Question and Expectation

Aschenbrenner, Dolich, Haskell, Macpherson and Starchenko proved in 2011 the following:

Theorem 4.14. $(\mathbb{Z}, +, 0, <)$ has no dp-minimal expansions.

Question 4.15. What happens with $(\mathbb{Z}, +, 0, |_p)$? Does it have dp-minimal expansions?

Question 4.16. Are there any other dp-minimal expansions of $(\mathbb{Z}, +, 0)$, unrealated to the ones with $<$ and to the ones with $|_p$?

Theorem 4.17 (Johnson)**.** Every dp-minimal field is ACF, RCF, or admits some valuation.

On this lines: is every dp-minimal expansion of $(\mathbb{Z}, +, 0)$ either stable (interdefinable with $(\mathbb{Z}, +, 0)$, interdefinable with $(\mathbb{Z}, +, 0, <)$, or interdefinable with $(\mathbb{Z}, +, 0, |_p)$.

4.2 Nicholas Wentzlaff – A Generalized Zariski Structure for Commutative C*-Algebras

Since we will speak of duality, we will first see a sketch of what happens in algebraic geometry, but from an unusual perspective.

Let $k \vDash \mathsf{ACF}$, and consider $A = k[x]/(f)$, where x is a tuple of variables and f is an irreducible polynomial, corresponding to a variety $V = \mathrm{Spec}(A)$.

Let $\mathbb{E}_A$ be the structure $(V, k, (a\colon V \to K)_{a\in A})$, where k is in the language of rings. The étale space $E(V, \mathcal{O}_V)$ is by definition the disjoint union $\bigsqcup_{p\in \mathrm{Spec}\, A} \mathcal{O}_{V,p}$. Except for $p = (0)$, these are the residue fields of the localisations A_p/pA_p, isomorphic to k, so $\bigsqcup_{p\in \mathrm{Spec}\, A} \mathcal{O}_{V,p} \cong \bigsqcup_{p\in \mathrm{Spec}\, A} k$. We have sections

$$\mathcal{F}(\mathcal{U}) = \{s\colon \mathcal{U} \to E \mid \text{continuous}, \ p \circ s = 1\}$$

$\mathcal{F}(V) = A$ is a topological space containing all the algebraic information. Consider the structure (V, R_φ), where φ is a positive, quantifier free formula in $\mathbb{E}_A$. On V^k this comes from $A^{\otimes^n}$. In other words, the positive definable sets are the Zariski closed ones.

If k is a valued field and V is a Berkovich space, Hrushovski and Loeser got topological information on V via model theory.

Gelfand-Naimark (commutative) duality: compact Hausdorff spaces X correspond to C*-algebras $A = \mathcal{C}(X)$, i.e. the set of continuous functions $X \to \mathbb{C}$. For this A, $\mathrm{Spec}(A) = \mathrm{SpecMax}(A) = \mathrm{char}(A) = \{f\colon A \to \mathbb{C}\}$ (multiplicative characters).

The weak* topology is the coarsest such that all evaluations at points are continuous.

The étale structure here is $\mathbb{E}_A = (X, \mathbb{C}, \alpha\colon X \to \mathbb{C})$, where now $\mathbb{C}$ has the ring structure plus the norm (in particular, $\mathbb{R}$ is definable). Consider $\chi(A) = (X, R_\varphi)$, where φ varies among the positive, quantifier free definable subsets of X in $\mathbb{E}_A$. Also, on X^n relations from $\chi(A^{\otimes^n})$.

Given a compact Hausdorff space X, not all closed subsets will be positive, quantifier-free definable, anyway:

Theorem 4.18. The sets $R \in \mathbb{R}$ give a subbasis of closed sets for the compact Hausdorff topology on $X^{(n)}$.

Proposition 4.19. If X is first-countable, then for all x we have $\{x\} \in \mathcal{R}$.

Proof. Every singleton is a G_δ-set, and

Lemma 4.20. Closed G_δ-sets are positively definable (so are in $\mathcal{R}$).

□

Proposition 4.21. If X is perfectly normal[1], then every closed set R is in $\mathcal{R}$.

Fact 4.22. These space are perfectly normal:

- Second countable
- Metrisable
- Topological manifolds

Theorem 4.23. If X is perfectly normal, then $(X, \mathcal{R})$ has quantifier elimination.

Theorem 4.24 (Scholium)**.** If X is any compact Hausdorff space, then $(X, \mathcal{R})$ has quantifier elimination, where $\mathcal{R}$ is the collection of all closed sets in all finite powers.

The proof, once you know what to show, is very elementary (basically topology from the '50s).

Proof. If $Y \subseteq X^n$, and X is compact Hausdorff and Y is locally compact, then projections $\pi(Y)$ are locally compact. But it is not hard to see that Y is locally compact if and only if it is constructible (something here uses Uryshon's Lemma). □

What about the tameness of this structure?

Proposition 4.25. If X is perfectly normal, not totally disconnected, then $(X, \mathcal{R})$ defines the reals with the field structure.

[1] One of several equivalent definitions is: every closed set is G_δ.

Proof. These spaces are subspaces of the unit cube. □

Worse than this,

Proposition 4.26 (Hrushovski?)**.** If X is perfectly normal, infinite, and Y is a discrete, definable subset of X, then every subset $Y' \subseteq Y^n$ is $\emptyset$-definable.

Think about $X = [0,1] \supseteq \omega + 1$. If you remove the accumulation point this is $\mathbb{N}$. Anyway, it seems at the moment that discrete sets are the only obstruction to the structure being model-theoretically nice.

Fact 4.27. For a compact Hausdorff space Y TFAE

- being profinite
- being totally disconnected
- $\dim Y = \mathrm{ind}(Y) = \mathrm{Ind}(Y) = 0$
- $\mathcal{C}(Y)$ is AF
- $\mathcal{C}(Y) = \mathcal{C}_s(Y)$.

So the idea is: instead of putting all closed sets in $\mathcal{R}$, we consider the equivalence classes $[R]$, where

$$[R] = \{R' \mid \dim R \triangle R' = 0\}$$

Consider the structure $\tilde{X} = (X, [R]_R)$. Work in progress (some parts conjectural): if X is compact Hausdorff, then $\tilde{X}$ has quantifier elimination, NIP, and is not stable. If X is perfectly normal, then $\tilde{X}$ is o-minimal (order induced by the lexicographic order on the unit cube) (this is the most conjectural part). (Zilber suggests: maybe "definable in an o-minimal structure"?) [some details being omissed, including something related to cylindric algebras]

Idea for NIP: if $a \in \mathcal{U}$, $\mathrm{tp}(a) \vDash [R](a)$, since the space is sober $R = R_1 \cup R_2$, so we have a chain $[R] > [R_1] > \ldots$ that reaches a point. So if $(a_i), b \in \mathcal{U}$, $[R](x,y)$ then $[R](x,b) = [R(x,b)]$ is constructible, so it cannot be true for a_i iff i is even.

4.3 Petr Glivický – Linear Fragments of Peano Arithmetic

[slides]

Presburger arithmetic is the full-induction arithmetic for the language $\{0, 1, -, +, \leq\}$, Peano arithmetic is the full-induction arithmetic for the language $\{0, 1, -, +, \cdot, \leq\}$. Linear arithmetic is the same thing for $\{0, 1, -, +, \cdot a \leq$

}, where $\cdot a$ is multiplication by an element a. Similarly, you can define LA_κ, i.e. with multiplication with κ elements. LA_0 is presburger, and LA_1 is LA.

Models M of LA can be equipped with a structure of a discretely ordered module over the ring $R_a = \mathbb{Q}[a] \cap M$: multiplication by any polynomial in a can be defined in the obvious way. The only thing to take care of is things like: if a is odd then $(a/2) \cdot x$ is not defined at 1. Analogously, models of LA_κ can be understood as discretely ordered modules over $\mathbb{Q}[a_\alpha \mid \alpha < \kappa] \cap M$.

How do definable sets in these structure look like? Two simpler related situations are unordered modules (forget the ordering), or forget the scalars (Presburger).

Theorem 4.28 (Baur-Monk)**.** Formulas in modules are equivalent to boolean combination of primitive positive formulas (pp-formulas), i.e. of the form $\exists z \psi_i$, where ψ_i is a system of linear equations.

For models of Presburger arithmetic,

Theorem 4.29 (Presburger)**.** Every formula in Presburger is equivalent to a disjunction of pp-formulas (now the ψ are linear *inequalities*).

In both cases there is "pp-quantifier elimination".

Question 4.30. Does this generalise to LA_κ for $\kappa > 0$?

Theorem 4.31. The answer is yes if and only if $\kappa = 1$.

As we said, think of $M \vDash \mathsf{LA}$ as a discretely ordered R_a-module.

We have $\mathbb{Z}[x] \subseteq R_a \subseteq \mathbb{Q}[x]$ and each R_a is up to iso uniquely determined by the class τ of remainders of a modulo $0 < n \in \mathbb{N}$ (so, modulo standard numbers). Denote R_τ some R_a with a having the remainders τ. Then, M is integrally divisible over R_a, i.e. for all $m \in M$ and $r \in R_a$

$$\exists x, z \in M \; m = rx + z \wedge 0 \leq z < r$$

Let $D = R_\tau$ be fixed and $M = \langle M, 0, 1, -, +, \leq, r \rangle_{r \in D}$ be a fixed discretely ordered (1 being the least positive) integrally-divisible D.module. In particular, any $M \vDash \mathsf{LA}$ with $D = R_a$ satisfies these assumption. Let M' be the expansion of M be definitions of functions r^{-1} providing integral division bu all scalars $0 < r \in D$.

Theorem 4.32. For any formula $\varphi(x, \bar{y})$ in M' there are finitely many terms $t_i(\bar{y})$, for $i < n \in \mathbb{N}$, such that

$$M' \vDash \exists x \; \varphi(x, \bar{y}) \leftrightarrow \bigvee_{i<n} \varphi(t_i(\bar{y}), \bar{y})$$

Corollary 4.33. M' has quantifier elimination.

Corollary 4.34. In M, every formula is equivalent to a disjunction of primitive positive formulas (just expand the definitions of the new symbols in M').

From now on, work in M' (so terms and formulas are M'-terms and M'-formulas). Denote with C the set of all constant terms in M'.

Call a term $t(\bar{x})$ harmonic if for some $q_i, r_i \in D$, $c \in C$ and $f\colon N \to \ell(\bar{x})$ we have

$$t(\bar{x}) = \sum_{i=0}^{N-1} q_i r_i^{-1}(x_{f(i)}) + c$$

(basically, a linear combinations of "stair functions" r_i^{-1}). Say that a formula or a piece-wise-term are harmonic iff all its maximal subterms are.

Call a piecewise-term τ an almost-term if it is of the form

$$\tau(\bar{x}) = \{s(\bar{x}) + c \text{ if } \psi_i(\bar{x}), i < n$$

where $s(x)$ is a term and $c \in C$.

Theorem 4.35 (Harmonic form theorem)**.**

1. Every term can be put in harmonic form modulo making it an almost-term
2. Every formula can be put in harmonic form.

Denote with $K(\bar{a})$ the ℓ-dimensional box with orthogonal edges of lengts $a_i + 1$, $i < \ell$, i.e. $\prod_{i<\ell}[0, a_i]$.

Theorem 4.36. Every set $A \subseteq M^n$ X-definable in M' (for $X \subseteq M$) can be written as $A = \bigcup_{i<k} g[P_i]$, where $g\colon (K(\bar{a}) \times M)^n \to M^n$ is a "linear coordination" of M^n, $\bar{a} = (a_0, \ldots, a_\ell) \in M^\ell$, $\ell \in \mathbb{N}$, and P_i, for $i < k$, are finitely many polyhedra in $(K(\bar{a}) \times M)^n$ over parameters from X.

In other words, "every definable set is a finite union of linear images of polyhedra".

Theorem 4.37 (Properties of LA)**.**

1. LA is model complete
2. For $A, B \models \mathsf{LA}$, it is $A \equiv B$ iff $a^A \equiv a^B \pmod{n}$ for all $0 < n \in \mathbb{N}$. [something on LA_τ]

something on R_τ

3. LA is decidable. LA_τ is decidable if and only if τ is recursive
4. THe induction shceme in LA may be equivalently replaced by the schem of [integral divisibility?]

Corollary 4.38. Up to elementary equivalence, models of LA are exactly ultraproducts of $\prod_{n\in\mathbb{N}}\langle\mathbb{Z},0,1,+,-,\underline{n},\leq\rangle/\mathcal{U}$

Remark 4.39. Properties of LA and Presburger are similar, but the proof for LA is way harder (due to [in]decomposability of sets $\{qx+ry \mid 0\leq x,y\in A\}$ in $A\vDash \mathsf{Pr}[\mathsf{LA}]$). In other words, the point is that there is some periodicity after you hit the lcm, but in LA this can be infinite.

Corollary 4.40. If M is a saturated model of Peano arithmetic, $0\leq a\in M\setminus\mathbb{N}$, $c,d\in M$, then TFAE:

1. For every "Peano multiplication " $\circ$ on M with $a\cdot x=a\circ x$ for all x, it is $c\cdot d=c\circ d$
2. $c\in\mathbb{Q}[a]$ or $d\in\mathbb{Q}[a]$.

In other words, if a multiplication is decided on a vertical line, then the only other decided multiplications are the ones with those c and d.

Theorem 4.41. LA is NIP but not dp-minimal.[2]

Proof Sketch. For NIP use that linear inequalities have VC-dimension 1; use the description as linear images of finite union of polyhedra. For non dp-minimality: $x\mapsto()$ [probably quotient and remainder] is a bijection between $[0,a)^2$ and $[0,a^2)$. ☐

Proposition 4.42 (with Pudlák)**.** There is a model of LA_2 and a non-standard $\ell\in M$ such that $\cdot\restriction[0,\ell]^2$ is definable in M for some Peano multiplication on M.

Proof Idea. In a saturated model of Peano arithmetic, for any $\mathbb{N}<\ell\in M$, we find elements a,b, such that the sequence $(1,1^2,2,2^2,\ldots,2\ell,(2\ell)^2)$ is encoded by the set of all numerators of convergents of the continued fraction of a/b. The crucial observation is that this set is definable in the language of LA_2 with a, b as the two scalars. ☐

Corollary 4.43. For $\kappa\geq 2$, LA_κ does not have pp-quantifier elimination (it is not even model-complete).

Question 4.44 (Chernikov and Hils)**.** Is every ordered module NIP?

Corollary 4.45. The model M from the previous proposition, considered as an ordered module over [?] is not NIP.

Question 4.46. Is there a non-trivial model (i.e. with $R=\mathbb{Z}[a,b]\cap M$ not any R_τ) of LA_2 which has pp-elimination/is model complete/is NIP.

[2]Presburger is dp-minimal.

4.4 Carlos Alfonso Ruiz Guido – Towards a Model Theory of Algebraic Stacks

A more precise title would be *Definability of Some Algebraic Stacks*. We will be vague about the definition of *stack*.

4.4.1 Motivation

Suppose that X is a smooth algebraic varieties over some $k \vDash \mathsf{ACF}$, an suppose we have an action $G \curvearrowright X$. It is not in general true that X/G is an algebraic variety (in particular, if the action is not free, it will not be one). On the other hand, if G and $\curvearrowright$ are definable, then X/G is definable, since ACF eliminates imaginaries.

There is a bigger category called k-algebraic stacks, where this quotient $[X/G]$ always exists. The real definition is quite involved; stacks are, instead of being sets with some sheaf, categories.

Question 4.47. From the model-theoretic point of view, what can we say about these objects, say in the simplest case $[X/G]$ where X is an algebraic variety?

These results are in very specific cases, but it could be possible to do it more generally.

4.4.2 Algebraic stacks

Definition 4.48. A stack is a category C together with some functor $F\colon C \to$ (g-alg. var) such that for any k-algebraic variety V, $F(V) := \{c \in C \mid F(c) = V\}$ with morphisms between $c \to c'$ being $\{f \in \mathrm{Mor}(c, c') \mid F(f) = \mathrm{id}_V\}$ (but for some reason the traditional notation is $F(V)$) is a groupoid[3]. (all arrows that get mapped to the identity of V) satisfying:

(a) ...

(b) ...

We will not see what those conditions are.

Definition 4.49. An k-algebraic stack is a stack such that there is some k-algebraic variety X (we will call it C_X; maybe we will see the definition of C_X later[4]) and a morphisms (think of it as an atlas) $p\colon C_X \twoheadrightarrow C$ such that p is surjective and étale.

[3] A small category where all morphisms are invertible.

[4] That was condition b); actually C_X is the category of pairs $(f\colon Y \to X, Y)$, where arrows are things that make triangles commute, and $F\colon C_X \to k$-algebras is the forgetful functor.

This is just to give an idea of why algebraic stacks are close to quotients.

Proposition 4.50. Given an algebraic stack C, consider $C_X \times_C C_x$ (fiber product), which is contained in $C_X \times C_X$ (think of this as X^2), defines a groupoid on C_X, i.e. with maps the projections. And $C(k)(= F^{(-1)}(k)) = X/\sim$, where $\sim$ is some equivalence relation coming from the groupoid (coars moduli space).

[Zilber: so these categories are useful to remember, after you take quotients, that your points have structure] The stack remebers the actions on the points.

4.4.3 Zariski Geometries

Definition 4.51 (Not the full one)**.** M is a Zariski geometry iff there is a family of topologies τ_n on M^n, together with some compatibility between the τ_n (e.g. projections continuous) and the [Krull?] dimension.

Example 4.52. If X is an algebraic variety, $X(k)^n$ with the Zariski topology.

Example 4.53 (Non-algebraic Zariski geometries)**.** Suppose X is an algebraic variety, and suppose we have $H \curvearrowright X(k)$. Suppose we have

$$0 \to K \to G \to H \to 0$$

where K is finite and the sequence does not split. Define $D := \bigcup_{x \in X/H} G \cdot x$. Consider the map $D \xrightarrow{p} X$ sending $\mapsto \pi(g) \cdot x$. Since X has a topology, we can consider the pullback topology on D. This makes D a Zariski geometry (not immediate to show).

Theorem 4.54 (Hrushovski, Zilber)**.** There are X, H, G such that D interprets a field (which will be algebraically closed), but non constructibly.

This Zariski geometry is not an object of algebraic geometry: it interprets a field, but it is not bi-intepretable with one.

More generally,

Theorem 4.55 (Zilber)**.** Constructed an infinite class of non-algebraic (not constructible in an ACF) Zariski geometries coming from non-commutative algebras.

Theorem 4.56 (Sustretov)**.** Found a definable groupoid for all those non algebraic Zariski geometries.

4.4.4 Generalised Imaginaries

The classical duality is; $\emptyset$-definable equivalence relations correspond to imaginary sorts. What Hrushovski did is: $\emptyset$-definable groupoids are in correspondence with "Generalised imaginary sorts"[5]. Some of them are not quotients, but torsors over [...]

Theorem 4.57 (Sustretov)**.** Suppose that M is a non algebraic Zariski geometry. There is some algebraic variety M $[V?]$ such that M is some generalised imaginary on M_V.

Theorem 4.58 (Ruiz Guido)**.** If M is a non algebraic Zariski geometry, given an étale map $U \to M_V$, then $C_M(U)$ (the groupoid over U) is definable in M.

[5]For those who know, they are internal covers.

Chapter 5

Contributed Talks – 27/01/17

5.1 Nadav Meir – Infinite Products of Ultrahomogeneous Structures

[slides]

Definition 5.1. A structure M is (elementarily) indivisible iff for every colouring of its universe in two colors, there is a monochromatic (elementary) substructure isomorphic to M.

Example 5.2. The countable model of DLO, the model companion of linear orders: if you color in red and blue either you have a blue open interval or a red dense set.

Example 5.3. The countable random graph, the model companion of simple graphs

Example 5.4. The countable ultrahomogeneous model ofthe rainbow graph: the model companion of the theory of infinitely many disjoint symmetric, irreflexive relations. Equivalently, the Fraïssé limit of all infinite-edge-colored finite graphs.

There are Fraïssé limits which are not indivisible: fields, for example. What about relational languages?

Example 5.5. The Fraïssé limit of finite bipartite graphs is not indivisible

Question 5.6. How rich is $\mathrm{Aut}(M)$, for M elementarily indivisible?

Question 5.7 (Hasson, Kojman, Onshuus)**.** Is there a rigid[1] elementarily indivisible structure? Is an elementarily indivisible structure homogeneous?

Here we mean

[1] I.e. the automorphisms group is trivial.

Definition 5.8. M is κ-ultrahomogeneous iff for every $A, B \subseteq M$ of cardinality $< \kappa$, every elementary map $A \to B$ extends to M.

In the rainbow graph, let

$$S = \{(x_1, x_2, y_1, y_2) \mid R_i(x_1, x_2) \wedge R_j(y_1, y_2) \mid i < j\}$$

this is ultrahomogeneous because has the same automorphisms group as the rainbow graph.

Let M, N be structures in a relational language L. The lexicographic product $M[N]$ is the L-structure on $M \times N$ where for any n-ary relation $R \in L$ we set

$$R^{M[N]} =$$

[missed it]

There is another notion called *generalised product* $M[N_a]_{a \in M}$. [and expansion $M[N_a]^s_{a \in M}$ with an equivalence relation]

Theorem 5.9 (Meir)**.** Let T_1, T_2 be theories ina relational language L admitting quantifier elimination and assume that T_1 has no non-trivial unary relations [probably the right hypothesis here is that T_1 has only one 1-type]. There is an $L \cup \{s\}$-theory T admitting quantifier elimination such that if $M \vDash T_1$ and $N_a \vDash T_2$ for all $a \in M$, then $M[N_a]^s_{a \in M} \vDash T$.

The proof is constructive (given quantifier elimination for T_1 and T_2). [proof sketch]

The s is essential:

Example 5.10. Consider $\Gamma[\Gamma]$ where Γ is the random graph. [explanation]

Corollary 5.11. If M and N are elementarily indivisible then $M[N]^s$ is as well.

Corollary 5.12. If $M \prec M'$ and $N_a \prec N'_a$, then $M[N_a]^s_{a \in M} \prec M'[N'_a]^s_{a \in M}$

Corollary 5.13. If T_1 and T_2 are complete, then T is complete. Same happens with model completeness, stability, NIP, equationality...

The corollary above follows from a normal form result [not written down]

Let M_0, M_1 be two elementarily indivisible structures such that $M_0 \prec M_1$ and $M_0 \ncong M_1$. If X is just an infinite set with no structure then $X[M_1]^s$ is elementarily indivisible. But what if one of those copies would be M_0? It is still elementarily indivisible! Same thing with n copies (violation of transitivity).

[...]

This goes in the direction of proving that $\mathrm{Aut}(M)$ is not rigid (proves weaker things). But:

The product can be iterated finitely many times easily. This corresponds somehow to a tree, so we can define a product of infinitely many products, induced by a tree, say $\omega^{<\omega}$ for simplicity. Also, assume each M_t is countable

Definition 5.14. Let $\langle M_t \mid t \in T \rangle$ be a family of structures in a relational language L indexed by T, such that the universe of M_t is suc(t). We define the product $\prod_{t \in T} M_t$ to be (a dense substructure of) the L-structure whose universe is the branches of T where for every k-ary relation $R \in L$ we set

$$R^{\prod_{t\in T} M_t} := \{(a_1, \dots, a_k) \mid M_m \vDash R(S_{a_1}(m), \dots, S_{a_k}(m)) \text{ where } m = a_1 \wedge \dots \wedge a_k\}$$

Where dense is in the tree sense.

We take a dense substructure because we want it to be countable.

Fact 5.15. In this infinite product there is no quantifier elimination. You need equivalence relations for all levels of the tree.

[example]

The problem in the example is that you cannot eliminate an infinite disjunction. So we use infinitary logic $\mathcal{L}_{\omega_1,\omega}$. You can define quantifier elimination in this language. A possible generalisation of the Theorem is [...] but for coutnable structures M, N, M admits $\mathcal{L}\omega_1,\omega$-QE iff M is ultrahomogeneous and $M \equiv_{\omega_1,\omega} N \iff M \cong N$ (Scott's Isomorphism Theorem. So actually the appropriate generalisation is

Theorem 5.16 (Meir). If $\langle M_t \mid t \in \omega^{<\omega} \rangle$ is a family of countable ultrahomogeneous structures in a countable language L such that $M_t \cong M_s$ whenever length(t) = length(s), then $\prod^{\bar{s}}_{t\in\omega^{<\omega}} M_t$ is ultrahomogeneous.

[proof sketch]

Corollary 5.17. There is a countable rigid elementarily infinitary structure.

[proof sketch]

There are generalisations of the last Theorem to structures of regular cardinality κ.

5.2 Jean-Cyrille Massicot – Around Hrushovski's Stabilizer Theorem

[slides]

Let X be a symmetric subset of a group G, i.e. $q \in X$ and $\forall x \in X\ x^{-1} \in X$

Definition 5.18 (Tao, 2005). Let $K > 0$ be an integer. X is a K-approximate subgroup iff there is a finite subset $E \subseteq G$ of cardinality $\leq K$ such that $X^2 \subseteq EX$.

Example 5.19. A 1-approximate subgroup is exactly a subgroup.

Example 5.20. In $(\mathbb{R}, +)$, the set $[-x, x]$ is a 2-approximate subgroup.

Theorem 5.21 (Breuillard, Green, Tao, 2012)**.** Let X be a finite K-approximate suggroup of G. There are L, d depending only on K and a K^6 approximate subroup $\tilde{X}$ L-equivalent to X such that $\langle \tilde{X} \rangle$ is virtually nilpotent of rank d.

The proof of this characterisation of approximate subgroups uses ultraproducts. It has two main steps: the first one is directly inspired by a paper of Hrushovski. The idea is to find a subgroup $H \subseteq X^4$ such that $\langle X \rangle / H$ is locally compact. Then you can construct a morphisms $\langle X \rangle / H \to \mathcal{L}$, where $\mathcal{L}$ is a Lie group.

In BGT, the construction of H uses a combinatorial theorem of Sanders, but the original paper of Hrushovski uses the *Stabiliser theorem.* The second step is inspired by the original proof of Gromov Theorem, coming from Gleason and Yamabe's solution of Hilbert's 5th problem.

Notation 5.22. Let $L = (X, 1, \cdot, ^{-1})$ be the language of approximate subgroups. Consider a language $L^* \supseteq L$ [...]

Definition 5.23. A set of formulas $\mathcal{I}$ is an ideal iff it is (nonempty and) stable under subsets and finite unions. In particular it contains $\emptyset$. A definable set $X \notin \mathcal{I}$ is said to be wide. A partial type is wide if it does not imply any formula in $\mathcal{I}$.

Definition 5.24. An ideal $\mathcal{I}$ is M-invariant if whether $\varphi(x, a) \in \mathcal{I}$ depends only on $\mathrm{tp}(a/M)$.

Definition 5.25. Let X be a definable set over M. Ana M-invariant $\mathcal{I}$ is S1 on X (over M) iff for any formula $\varphi(x, a) \subseteq X$ and M-indiscernible sequence (a_i) in $\mathrm{tp}(a/M)$ [...]

Example 5.26. If T is a simple theory, generic formulas can be defined using local ranks, and non-generic formulas form an ideal. One can show that this is an S1 ideal on any generic set.

Let p be a wide type Let $\mathrm{Stab}(p) = \langle \mathrm{St}(p) \rangle$, where $\mathrm{st}(p)$ is $\{g \in \langle X \rangle \mid gp \cap p \text{ is wide}\}$.

Theorem 5.27 (Stabilizer theorem, Hrushovski 2012)**.** Let $\mathcal{I}$ be an M-invariant ideal stable under left multiplication. Assume there is a wide symmetric definable subset X such that $\mathcal{I}$ is $S1$ on X^4. Let $p \subseteq X$ be any wide completion of X. Then $\mathrm{Stab}(p) = (pp^{-1})^2$ and it is a type-definable subgroup of bounded index in $\langle X \rangle$. Moreover, $\mathrm{Stab}\, p$ is the smallest M-type-definable subgroup of bounded index in $\langle X \rangle$. Write $\mathrm{Stab}(p) = X^{00}_M$. It is normal and does not depend on p.

The proof relies on the fact that the relation $R(a, b)$ defined as "$ap \cap bp$ is wide" is a stable relation.

If X is an approximate subgroup, then one has

Proposition 5.28. Let $\mathcal{I}$ be an M-invariant ideal stable under left multiplication. Assume there is an approximate subgroup X on which $\mathcal{I}$ is S1. Then $\mathcal{I}$ is S1 on X^n for all $n > 0$.

In this case we say that $\mathcal{I}$ is an S1 ideal.

Proposition 5.29. Under the assumptions of the Stabiliser theorem, X^4 is an approximate subgroup. In particular, $\mathcal{I}$ is S1 on X^n for all $n > 0$.

Definition 5.30. A definable approximate subgroup X is definably amenable iff there is a lef translation invariant finitely additive meaure μ on the L-definable subsets of $\langle X \rangle$ with $\mu(X) = 1$.

Let $\varphi(x, a_i)$ be an indiscernible family of subsets of X. Can we have simultaneously $\mu(\varphi(x, a_0)) > 0$ and $\mu(\varphi(x, a_0) \wedge \varphi(x, a_1)) > 0$. If in addition μ is M-invariant, that is impossible, which means that the ideal $\mathcal{I}$ of sets with measure 0 is S1. One can make the measure definable, thus invariant, by adding a predicate $R_{\varphi,\alpha}(y)$ for every $\varphi(x, y) \in L$ and $\alpha > 0$ for the set of bs such that $\varphi(x, y) \geq \alpha$. One obtains $L^* \supseteq L$ such that $\mathcal{I}$ is S1 in the corresponding structure.

The stabiliser theorem gives a normal subgroup $H = \operatorname{Stab} p$ in X^4 of bounded index in $\langle X \rangle$ and $L^*(M)$-type-definable. However there a better result in this case:

Theorem 5.31 (Massicot-Wagner '14)**.** Let X be a defianbly amenable approximate subgroup, then there is an $L(M)$-type-definable normal subgroup $H \subseteq X^4$ of bounded index in $\langle X \rangle$.

The proof is adapted from he combinatorial theorem of Sanders used in BGT, with only a minimal amount of model theory.

Theorem 5.32 (Massicot, '16)**.** Let X [...] Then there is an L-type-definable (normal) subgroup [...]

[Beth definability] Beth definability has an analogue for types in a single saturated model (working in a monster, you never have two models with the same domain). [kind of longish statement; it is in terms of L automorphisms which are not $L^*(\supseteq L)$-automorphisms]

Theorem 5.33 (type-definable version of Schlichting's Theorem, Wagner '16)**.** Let G be a definable group, $\mathcal{H}$ a type-definable family of commensurable subgroups, Γ a type-definable group of automorphisms of G stabilising $\mathcal{H}$ setwise. Then there is a Γ-invariant type-definable subgroup $\tilde{H}$ commensurable with the elements of $\mathcal{H}$. (not really, there are more hypothesis on the theory)

[Proof sketch of Massicot's Theorem]

(there may be some underlying set-theoretic assumptions because the version of Beth definability assumes a saturated model (as opposed to κ-saturated κ-strongly-homogeneous)

[applications to Stabiliser Theorem, under [hypotheses] gives a normal subgroup of X^8 of bounded index in $\langle X \rangle$ which is L-type definable without parameters.]

5.3 Thomas Kirk – Topological Dynamics and Model Theory

Definition 5.34. A semigroup $(X, \cdot)$ is an *Ellis semigroup* iff X is compact Hausdorff and for all b the function $f_b \colon x \mapsto xb$ is continuous.

Definition 5.35. $I \subseteq X$ is a *closed left ideal* iff it is topologically closed, a left ideal (absorbs on the left), and non-empty.

Theorem 5.36 (Ellis)**.** If $(X, \cdot)$ is an Ellis semigroup, I is a closed left ideal, and $J = \{a \in X \mid a^a = a\}$, then:

- $I \cap J \neq \emptyset$
- If I is minimal and $u \in I \cap J$, then $(uI, \cdot)$ is a group (Ellis group)
- All such groups, as I and u vary, are isomorphic.

Example 5.37. A G-flow is an action $G \times X \to X$ on a compact Hausdorff topological sapce X such that all the functions : $\pi_g \colon X \to X$ are homeomorphisms. The closure of $\{\pi_g \mid g \in G\}$ in X^X with the product topology is a semigroup $(E(X), \circ)$, that we call the *enveloping semigroup*. It is an Ellis semigroup.

$E(X)$ is itself a G-flow.

Theorem 5.38 (Ellis)**.** The minimal subflows of $G \curvearrowright E(X)$ coincide with the minimal left ideals.

Consider the space $S_G(M)$ of types over M concentrating on G. It can be made into a flow $(G(M), S_G(M))$, where the action is $(g, p) \mapsto \mathrm{tp}(g \cdot a/M)$, where $a \vDash p$. Then $E(S_G(M), \circ)$ is an Ellis semigroup.

Theorem 5.39 (Newelski)**.** If all types over M are definable, $(E(S_G(M)), \circ)$ is homeomorphic to $(S_G(M), *)$, where $p * q = \mathrm{tp}(a \cdot b/M)$ for $a \vDash p$ and $b \vDash q \mid (M, a)$ ($\mid$ indicates the coheir extension).

Let G^0_A be the smallest A-definable subgroup of finite index, and G^{00}_A be the smallest A-type-definable subgroups of bounded index. If these do not depend on A, we drop the subscript and say that "G^{00} exists".

Fact 5.40 (Shelah)**.** If T has NIP, then G^{00} exists.

Example 5.41. For $\mathrm{SO}_2(\mathbb{R})$, G^{00} is the infinitesimal subgroup. For a sufficiently saturated model M, we have $\mathrm{SO}_2(M)/\mathrm{SO}_2(M)^{00} \cong \mathrm{SO}_2(\mathbb{R})$.

Theorem 5.42 (Newelski)**.** If T is stable, then the minimal ideal of $(S_G(M), *)$ is precisely the generic types $S_{g,\mathrm{gen}}(M)$. Moreover, $uI = I$ and $I \cong G/G^{00}$ as topological groups.

(by stability, $G^{00} = G^0$). Newelski conjectured this was true in the NIP case, but it turned out to be false.

Definition 5.43. A global type $p \in S_G(\mathfrak{U})$ is f-generic over M iff $g \cdot p$ is $\mathrm{Aut}(\mathfrak{U}/M)$-invariant for all $g \in G(\mathfrak{U})$.

Theorem 5.44 (Hrushovski-Pillay)**.** If T is NIP, then $G(\mathfrak{U})$ is definably amenable if and only if there is some f-generic type over some small M. In this case, then $G^{00} = \mathrm{Stab}(p)$.

Example 5.45. Examples of definably amenable groups: solvable groups, definably compact groups in o-minimal theories, stable groups.

Theorem 5.46 (Pillay, Simon, Chernikov)**.** When G is definably amenable and T is NIP, $(uI, \cdot) \cong G/G^{00}$ when any of the following hold:

- G has fsg: there is a global type $p \in S_G(\mathfrak{U})$ and a small M such that the translates $g \cdot p$ are finitely satisfiable in M for all $g \in G$.
- G has a definable f-generic.
- (The first two cases cover every case definable in dp-minimal)
- G is definably amenable in an o-minimal theory.

The counterexample without definable amenability is

Counterexample 5.47 (Penazzi, Gismatullin, Pillay)**.** In $\mathbb{R}$, $\mathrm{SL}_2(\mathrm{SL}_2$ is trivial, but $(uI, \cdot)$ is the group with two elements.

Theorem 5.48 (Yao)**.** For $M \vDash$ RCF, if $G = HK$ where H is torsion-free and K is definably compact, then the Ellis group of $(S_G(M^{\mathrm{ext}}), *)$ is $N_G(H) \cap K(\mathbb{R})$, where N denotes the normaliser.

In the case above, K is $\mathrm{SO}_2(B)$ and H is B (the product is not direct). What about $\mathbb{C}((t))$?

5.4 Jan Hubička – Ramsey Properties of Hrushovski Construction

[slides]

Theorem 5.49 (Ramsey, 1930)**.** $\forall n, p, k \geq 1 \exists N\ N \to (n)^p_k$ [spells out]

Ramsey's theorem was originally intended to solve a problem in logic but was picked up for combinatorics.

Assume there is an order and the language is relational.

Theorem 5.50 (Nesetril-Rödl; Abramson-Harrington)**.** $\forall A, B \in \mathrm{Rel}(L) \exists C \in \mathrm{Rel}(L) C \to (B)^A_2$

$\binom{B}{A}$ is the set of all substructures of B isomorphic to A. $C \to (B)^A_2$: for every 2 colouring of $\binom{C}{A}$ there is $\tilde{B} \in \binom{C}{B}$ such that $\binom{\tilde{B}}{A}$ is monochromatic.

The order is necessary: if A is two points with an ordered edge and B is a cyclic triangle, then [. . .]

Definition 5.51. A class $\mathcal{C}$ of finite L structures is Ramsey iff $\forall A, B \in \mathcal{C} \exists C \in \mathcal{C}\ C \to (B)^A_2$.

Example 5.52. Class of all linear orders. This is basically Ramsey's theorem.

Example 5.53. All ordered structures over a fixed relational language (previous theorem)

Example 5.54. Finite partial orders with linear extension.

Example 5.55 (Hubička-Nešetřil, '16)**.** For every language, $\mathrm{Mod}(L)$.

Fact 5.56 (Nesetril, '80s)**.** Under mild assumptions Ramsey classes have amalgamation property.

Nesetril classification programm, 2005: (nice) Ramsey classes are amalgamation classes. This give homogeneous structures via limits and you can somehow come back to Ramsey classes via expansion of homogenous.

More recently: from homogenous strictures to universal minimal flows; expansion of homogeneous correspond to extremely amenable groups, and from this you have universal minimal flows.

Definition 5.57. Let $L' \supseteq L$. An expansion (or lift) is [. . .]

[. . .]

Question 5.58. Does every amalgamation class have a Ramsey expansion?

Answer: yes, extend language by infinitely many unary relations; assign every vertex to a unique relation.

Definition 5.59 (Nguyen van Thé)**.** Let $\mathcal{K}$ be a class of L-structures and $\mathcal{K}'$ a class of expansions of $\mathcal{K}$. $\mathcal{K}'$ is precompact wrt $\mathcal{K}$ iff for every $A \in \mathcal{K}$ there are only finitely any expansions of A in $\mathcal{K}'$. $\mathcal{K}'$ has the expansion property iff for every $A \in \mathcal{K}$ there is $B \in \mathcal{K}$ such that every expansion of B in $\mathcal{K}'$ contains every expansion of A in $\mathcal{K}'$.

[...Theorem ...]

Question 5.60. Does every amalgamation class have a precompact Ramsey expansion?

No: consider $\mathbb{Z}$ as a metric space.

Question 5.61 (Nguyen van Thé)**.** Does every ω-categorical structure have a precompact Ramsey expansion?

Theorem 5.62 (Evans, 2015+)**.** There is a countable, ω-categorical structure with no precompact Ramsey expansion.

This was done via Hrushovski constructions. Here we explore properties of this example. There are three variants: $\mathcal{C}_0$ is the easy one; $\mathcal{C}_1$ is the kindergarten one and $\mathcal{C}_F$ is the actual counterexample. We will see $\mathcal{C}_0$.

Define the predimension of a graph $G = (V, E)$ as $\delta(G) = 2|V| - |E|$. Say that a finite graph G is $\mathcal{C}_0$ iff for all $H \subseteq G$ we have $\delta(H) \geq 0$. Say that $G \subseteq H$ is self-sufficient, $G_{\leq}sH$ iff for all $G \subseteq G' \subseteq H$, $\delta(G) \leq \delta(G')$.

Lemma 5.63. $\mathcal{C}_0$ is closed for free amalgamation over self-sufficient substructures.

Proof. $\delta(C) = \delta(B) + \delta(B')_\delta(A)$ □

Lemma 5.64 (By marriage theorem)**.**

- $G \in \mathcal{C}_0$ iff it has 2-orientation (out-degrees at most 2)
- $H \leq_s G$ iff G can be 2-oriented with no edge from H to $G \setminus H$.

Corollary 5.65. $\mathcal{C}_0$ is a reduct of the class of all finite 2-orientations $\mathcal{D}_0$.

Theorem 5.66 (Kechris, Pestov, Todorcevic, '05)**.** Let F be a Fraissé limit. Then tfae:

- the automorphism group of F is extremely amenable
- Age(F) has the ramse property.

Let M_0 be the generalised Fraissé limit of $\mathcal{C}_0$. [...]

Theorem 5.67 (Evans 2016)**.** There is no precompact Ramsey expansion of $(\mathcal{C}_0, \leq_s)$.

[proof sketch]

Theorem 5.68 (Hubicka, Evans, Nesetril, 2015+)**.** $\mathcal{D}_0^{\prec}$ is a Ramsey class.

[proof sketch]

Question 5.69 (Tsankov)**.** Is $(\mathcal{D}_0^{\prec}; \sqsubset_s)$ any better than the trivial Ramsey expansion?

Theorem 5.70 (Hubicka, Evans, Nesetril 2016+)**.** There is $\mathcal{G}_0 \subseteq \mathcal{D}_0^{\prec}$ such that [. . .]

Definition 5.71. $\mathcal{K}'$ has the expansion property with respect to $\mathcal{K}$ iff for every $A \in \mathcal{K}$ there is $B \in \mathcal{K}$ such that every expansion of B in $\mathcal{K}'$ contains overy expansion of A in $\mathcal{K}'$.

Let $(\mathcal{D}_1, \sqsubset_s)$ the class of all finite acyclic orientations and with $(\mathcal{C}_1, \sqsubset_s)$ its unoriented reduct.

Theorem 5.72. FOr every $A^+ \in \mathcal{D}_1$ there is $B \in \mathcal{C}_1$ such that every expansion $B^+ \in \mathcal{D}_1$ contains A^+ as a self-sufficient substructure.

[proof sketch] [. . .]

Theorem 5.73 (Hubicka, Evans, Nesetril, '16+)**.** For every $A^+ \in \mathcal{E}_1$ there is $B \in \mathcal{C}_1$ such that every expansion $B^+ \in \mathcal{E}_1$ contains A as a self-sufficient substructure.

Given a strong class $(\mathcal{C}; \leq)$ a strong partial automorphism of $A \in \mathcal{C}$ is an isomorphism $f \colon D \to E$ for some $D, E \leq A$.

[definition: eppa for strong partial automorphisms]

Theorem 5.74 (Evans 2016, easier argument by Tsankov)**.** $\mathrm{Aut}(M_0)$ is not amenable and thus $(\mathcal{C}_0, \leq_s)$ has no eppa

Theorem 5.75 (Hubicka, Evans, Nesetril, '17+)**.** The class of finite 2-orientations closed for free amalgamation over successor-closed substructures has EPPA.

[proof sketch] [summary]

ω-categorical case: $F \colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$. Consider $\mathcal{C}_f = \{B \mid \forall A \subseteq B\ \delta(A) \geq F(|A|)\}$. Define $A \leq_d B$ iff $\delta(A) < \delta(B')$ (it is important that the inequality is strict) for all finite B' with $A \subset B' \subseteq B$.

Lemma 5.76. Put $F(x) = \log x$. Then $(\mathcal{C}_F, \leq_d)$ is a [amalgamation class?].

[. . .][. . .][. . .]