Orbital Stability of Standing-Wave Solutions to the Non-Linear Schrödinger Equation in dimension one

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When we search standing-wave solutions

$$\phi(t,x) = e^{i\omega t}u(x), \quad \omega \in \mathbb{R}, \quad u \in H^1(\mathbb{R}^n;\mathbb{R})$$

to the non-linear Schrödinger equation

$$i\partial_t\phi + \Delta_x\phi - g(\phi) = 0,\ (t,x) \in [0,+\infty) \times \mathbb{R}^n,\ \phi \in \mathbb{C}$$

we need to solve the equation $\Delta u(x) - g(u(x)) - \omega u(x) = 0$.

A pair (u,ω) can be obtained as a critical point to

$$E\colon H^1(\mathbb{R}^n;\mathbb{C}) \to \mathbb{R}$$

$$E(v) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v(x)|^2 dx + \int_{\mathbb{R}^n} G(v(x))dx, \quad G' = g$$

on the constraint $S(\lambda) = \{v \in H^1(\mathbb{R}^n;\mathbb{C}) \mid \|v\|_{L^2}^2 = \lambda\}$.

$$G_\lambda := \{v \in H^1(\mathbb{R}^n;\mathbb{C}) \mid v \in S(\lambda),\ E(v) = \inf_{S(\lambda)} E\}.$$

Stable subsets of $H^1(\mathbb{R}^n; \mathbb{C})$

Given $v \in H^1(\mathbb{R}^n; \mathbb{C})$ and $t \geq 0$, we define

$$U_t(v)(x) := \phi(t, x), \quad U_t \colon H^1 \to H^1$$

where ϕ solves the initial value problem

$$i\partial_t \phi(t, x) + \Delta_x \phi(t, x) - g(\phi(t, x)) = 0, \quad \phi(0, x) = v(x).$$

Definition (Stable subsets of $H^1(\mathbb{R}^n; \mathbb{C})$)

$S \subseteq H^1(\mathbb{R}^n; \mathbb{C})$ is stable if $\forall \varepsilon > 0 \; \exists \delta > 0$ such that

$$\operatorname{dist}(v, S) < \delta \Rightarrow \operatorname{dist}(U_t(v), S) < \varepsilon$$

for every $t \geq 0$ and $v \in H^1(\mathbb{R}^n; \mathbb{C})$.

The equality $E(zv(\cdot + y)) = E(v)$ for every $(z, y) \in S^1 \times \mathbb{R}^n$ gives

$$G_\lambda(u) := \{zu(\cdot + y) \mid z \in S^1,\ y \in \mathbb{R}^n\} \subseteq G_\lambda.$$

If $G_\lambda(u)$ is stable, we say that $u(x)e^{i\omega t}$ is *orbitally stable*.

V. Benci *et al.* 2007, Adv. Nonlinear Studia

The set G_λ is non-empty and stable, provided

$$\text{(gc)} \quad |g(s)| \leq c(|s|^{p-1} + |s|^{q-1}), \quad 2 < p < 2 + \frac{4}{n}, \quad \inf(G) < 0.$$

(gc) is considered the minimum requirement to have G_λ.

It follows from the Concentration-Compactness Lemma (Lions).

We are interested on the stability of $G_\lambda(u)$ when (gc) is satisfied.

The pure power case

T. Cazenave and P. L. Lions, CMP, 1982, $n \geq 1$

If $g(s) = -a|s|^{p-1}$ (pure power), then $G_\lambda(u)$ is stable.

The main ingredients, are the symmetry $g(ts) = t^{p-1}g(s)$ and

M. K. Kwong, ARMA 1989

Given $\omega > 0$, there is only one solution $u \in H^1_{r,+}$ to

$$\Delta u + au^{p-1} - \omega u = 0.$$

By definition, $u \in H^1_{r,+}$ if u is real valued, symmetric, non-negative.

In their paper, Cazenave and Lions proved that $G_\lambda(u) = G_\lambda$.

Byeon, Jeanjean and Mariș, Calc. Var., 2009

$G_\lambda(u)$ has a unique positive and symmetric representative in $H^1_{r,+}(\mathbb{R}^n)$.

If we define

$$P := G_\lambda \cap H^1_{r,+}$$

then T. Cazenave and P.L. Lions proved that $\#P = 1$.

Lemma (G. and Georgiev, (gc))

If P is finite, then $G_\lambda(u)$ is stable for every $u \in G_\lambda$.

Kwong's result has been extented by Serrin and Tang (IUMJ 2000).

However,

$$g(s) = -a|s|^{p-1} + b|s|^{q-1}, \quad a, b > 0.$$

does not satisfy their requirement. And symmetry fails.

So far, we have a weak uniqueness result

G. and Georgiev, $n \geq 1$

g is C^1 and $g(0) = 0 = g'(0)$. If $u_1, u_2 \in H^1_{r,+}$ solve

$$\Delta u - g(u) - \omega u = 0,$$

and $\|u_1\|_2^2 = \|u_2\|_2^2$, then $u_1 = u_2$.

We do not know whether $E(u_1) = E(u_2) \Rightarrow \omega_1 = \omega_2$.

The dimension $n = 1$

Conjecture 1, $n = 1$

We think that the functional E over $S(\lambda) \cap H^1_r$ is non-degenerate.

Then $P = G_\lambda \cap H^1_r$ is a finite set.

It was proved by M. Weinstein (CMP, 1986) for solutions to

$$u'' - g(u) - \omega u = 0$$

under the additional a-priori assumptions on u

$$\int_{-\infty}^{+\infty} \left(\frac{g(u(x))}{u(x)} + \left(g'(u(x)) - \frac{g(u(x))}{u(x)} \right) u'(x)^2 \right) dx \neq 0.$$

We wish to remove this assumption.

Conjecture 2, $n = 1$

If E is non-degenerate on $S(\lambda) \cap H^1_r$, then $\#P = 1$.

Pure-power non-linearities enjoy special rescalings.

Given $\omega > 0$, if u solves

$$\Delta u - \omega u + |u|^{p-2} u = 0$$

then

$$u(x) = \omega^{1/(p-1)} u_0(\omega^{1/2} x), \quad \|u\|_{L^2}^2 = \omega^{\frac{2}{p-1} - \frac{n}{2}} \|u_0\|_{L^2}^2.$$

So, the solution is unique for every ω.

If $\|u\|_{L^2}$ is λ, there is only the pair

$$\left(\omega^{1/(p-1)} u_0(\omega^{1/2} x), \omega\right)$$

where

$$\omega = (\lambda \|u_0\|_{L^2}^{-1})^{2\alpha}, \quad \alpha := \frac{2(p-1)}{4 - n(p-1)}.$$

Serrin and Tang (IUMJ, 2000) generalized Kwong's result.

However, they require

$$2G(s) + \omega s^2$$

to have a unique zero.

Berestycki and Lions • Arch. Ration. Mech. Anal., 1983 • $n = 1$

If the first positive zero of $2G + \omega s^2$ is simple, then the solution to

$$u'' - g(u) - \omega u = 0$$

is unique.

If the H^1 is replaced by (V), the uniqueness fails:

Del Pino, Guerra, Davila • Proc. Lond. Math. Soc., 2013 • $n = 3$

For every $1 < p < 3$, there exists (a, q) such that

$$\Delta u - u + u^p + au^q = 0$$

has at least three solutions in $H^{1,+}_{loc,r} \cap V$.