On the compactness of minimizing sequences of an energy functional arising from a system of Non-Linear Klein-Gordon Equations

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Concentrated-compactness of minimizing sequences

We are given the two functionals

$$E, C\colon H^1(\mathbb{R}^N; \mathbb{R}^k) \to \mathbb{R}$$

and the constraint

$$M_\sigma = \{u \in H^1 \mid C(u) = \sigma\}.$$

A minimizing sequence (u_n) of (E, M_σ) exhibits a concentration-compactness behaviour if there exists $(y_n) \subseteq \mathbb{R}^N$ and $u \in H^1$ such that

$$u_n(\cdot + y_n)(x) := u_n(x + y_n)$$

converges to u in H^1. If this happens for every minimizing sequence, we say that $\sigma \in \Omega \subseteq \mathbb{R}$.

From concentration-compactness of the minimizers of (E, M_σ) it follows the existence and orbital stability of standing-waves solutions to

(1) T. Cazenave and P. L. Lions, NLS, $N \geq 3$, 1982,

(2) Z. Wang, N. V. Nguyen, 2-NLS, 3-NLS, $N = 1$, 2011 and 2013

(3) NLS + KdV (J. Albert and J. Angulo Pava, $N = 1$, 2003)

(4) NLKG (J. Shatah and W. Strauss, $N \geq 3$, pure powers, 1985)

(5) NLKG (V. Benci, C. Bonanno *et al.*, $N \geq 3$, 2010, general non-linearities)

(6) 2-NLKG (G., $N \geq 3$, 2012)

(7) NLS + KdV (J. Albert and S. Bhattarai, $N = 1$, 2013).

(8) V. Benci and D. Fortunato, $N \geq 1$, 2014, general devices for several equations.

Systems of non-linear Klein-Gordon equations

Given $1 \leq k$, a system of NLKG equation is

$$(k\text{-NLKG}) \qquad \partial_{tt} v_i + m_i^2 v_i + \partial_{z_i} G(v) = 0, \quad 1 \leq i \leq k$$

where

$$v(t, \cdot) \in H^1(\mathbb{R}^N; \mathbb{C})$$

for every t and

$$0 < m := m_1 \leq m_2 \leq \cdots \leq m_k.$$

If the standing-wave v

$$v_i(t, x) := u_i(x) e^{i\omega t}, \quad (u, \omega) \in H^1(\mathbb{R}^N; \mathbb{R}^k) \times \mathbb{R}^k$$

solves (k-NLKG), then u solves the elliptic problem

$$-\Delta u_i + (m_i^2 - \omega_i^2) u_i + \partial_{z_i} G(u) = 0.$$

The variational setting

In turn, a solution of the elliptic problem can be obtained as a critical point of

$$E\colon H^1(\mathbb{R}^N;\mathbb{R}^k)\times\mathbb{R}^k\to\mathbb{R}$$

$$E(u,\omega) := \frac{1}{2}\int_{\mathbb{R}^N}\left(\sum_{i=1}^k |\nabla u_i|^2 + (\omega_i^2 + m_i^2)u_i^2 + 2kG(u)\right)$$

on the constraint

$$M_\sigma := \left\{(u,\omega)\in H^1\times\mathbb{R}^k \mid C(u,\omega)=\sigma\right\}$$

where

$$C(u,\omega) = \sum_{i=1}^k \omega_i \int_{\mathbb{R}^N} u_i^2.$$

Assumptions on G

- G is continuous
- there are $2 < p \leq q$ such that

$$|G(z)| \leq c(|z|^p + |z|^q);$$

  in the case $N \geq 3$, we assume that $q < \frac{2N}{N-2}$ too

- $$F(z) := G(z) + \frac{1}{2}\sum_{i=1}^{k} m_i^2 |z_i|^2 \geq 0$$

(1) For which values of σ, we have $\sigma \in \Omega$?

(2) if (u, ω) is a minimum, for which i's $u_i \neq 0$?

We define

$$\beta_K := \left(2 \inf_{z\neq 0} \frac{F(z)}{|z|^2}\right)^{1/2} \quad \mu := \left(2 \liminf_{z\to 0} \frac{F(z)}{|z|^2}\right)^{1/2} = m$$

and

$$I(\sigma) := \inf_{M_\sigma} E \quad L(\sigma) := \frac{I(\sigma)}{\sigma}.$$

In general, $\beta_K \leq \mu$.

L is non-negative, strictly decreasing and $\inf(L) = \beta_K$.

Theorem

(i) *If $\beta_K = \mu$, then E does not achieve its infimum on M_σ*

(ii) *if $\beta_K < \mu$, then $\sigma \in \Omega$ if $L(\sigma) < \mu$.*

In case (ii), the set $\Omega \neq \emptyset$.

We define

$$K := \{1, 2, \ldots, k\}.$$

Given $J \subseteq K$

$$\Sigma(J) := \{z \in \mathbb{R}^k \mid i \notin J \implies z_i = 0\}$$

$$\beta_J := \left(\inf_{z \in \Sigma(J)} \frac{F(z)}{|z|^2} \right)^{1/2}$$

if $J \neq \emptyset$, and $\beta_\emptyset := \mu$.

For every $1 \leq m \leq k$, we define

$$\gamma_m := \min\{\beta_H \mid \#H = m\}.$$

So, $\gamma_0 = \mu$ and $\gamma_m \leq \gamma_{m-1}$.

We assume that $\beta_K < \mu$ and $L(\sigma) < \mu$. So $\sigma \in \Omega$.

Theorem

Let u be a minimum of E over M_σ and $1 \leq m \leq k$. If

$$L(\sigma) < \gamma_{k-m}$$

there are at least $k - m + 1$ non-trivial components.

In particular, all the components of u are non-trivial if $L(\sigma) < \gamma_{k-1}$.

The case $L(\sigma) < \gamma_{k-1}$: minima are completely non-trivial

If $(u, \omega) \in M_\sigma$ is a minimum.

$$u_i = 0 \implies u(x) \in K - \{i\}$$

$$L(\sigma) = \frac{E(u,\omega)}{C(u,\omega)} \geq \inf_{\omega} \frac{E(u,\omega)}{C(u,\omega)}$$

$$= \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + 2 \int_{\mathbb{R}^N} F(u)}{\|u\|_{L^2}^2} \right)^{1/2} \geq \beta_{K-\{i\}} \geq \gamma_{k-1}$$

We obtain a contradiction with $L(\sigma) < \gamma_{k-1}$.

Proof: the vanishing case

Let (u_n, ω_n) be a minimizing sequence for (E, M_σ).

Suppose that

$$u_n^i(\cdot + y_n) \rightharpoonup 0$$

for every $1 \leq i \leq k$ and every sequence (y_n). Then, by Lemma I.1 (P. L. Lions, 1984)

$$\|u_n\|_{L^p(\mathbb{R}^N)} \to 0 \quad 2 < p < \frac{2N}{N-2}.$$

Then

$$L(\sigma) \simeq \frac{E(u_n, \omega_n)}{C(u_n, \omega_n)} \geq \left(\frac{2 \int_{\mathbb{R}^N} F(u_n)}{\|u_n\|_{L^2}^2} \right)^{1/2} \simeq \left(\frac{\sum_{i=1}^k m_i^2 \|u_n^i\|_{L^2}^2}{\|u_n\|_{L^2}^2} \right)^{1/2} \geq m = \mu.$$

The dichotomic case

For every $\tau \leq \sigma$

$$L(\sigma) \leq L(\tau)$$

and equality holds if $L(\tau) = L(\sigma) = m$ (rescaling argument).

If (u_n) does not vanish, there exists (y_n) such that

$$u_n(\cdot + y_n) \rightharpoonup u$$

such that $u \neq 0$. If

$$\tau := C(u, \omega) < \sigma$$

we obtain a contradiction.

Final remarks and extensions

(1) In [5], when $k = 1$ and $q < 2 + 4/N$, $\Omega = (0, +\infty)$

(2) in [8], less is known about Ω

(3) we prove that $\Omega := \{L < m\}$

(4) we expect solutions with trivial components if $\gamma_m < L(\sigma) \leq \gamma_{m-1}$

(5) the critical case $q = 2N/(N-2)$: sequences as

$$u(\cdot + y_n) + R_n^{N/2} v(R_n(\cdot + z_n))$$

may arise.