EXERCISES FOR ORAL EXAMS

Exercise 1. Is the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}$ injective or surjective? Show that it is locally invertible.

Exercise 2. In the linear space of polynomials of degree $\leq n$, $\mathbb{R}_n[X]$, you can define the linear map

$$T\colon p \mapsto e^{-X} \int e^t p(t) dt.$$

Prove that $T(\mathbb{R}_n[X]) \subseteq \mathbb{R}_n[X]$ and that is it invertible. [1]

Exercise 3. Let f_1 and f_2 two linear application on a space V. Find necessary and sufficient conditions which ensure the existence of a linear application $g \neq 0$ such that

$$g \circ f_1 = f_1 \circ g = g \circ f_2 = f_2 \circ g = 0$$

Exercise 4. Let V be an $\mathbb{R}$-linear space of finite dimension. Prove that the set $GL(V)$ of linear invertible maps generates the linear space of linear maps $\mathscr{L}(V)$. [2]

Exercise 5. Give an example of linear space X and function $f : X \to X$ such that the sequences $\ker(f^i)$ ed $\mathrm{Img}(f^i)$ are not stable.

Exercise 6. In $\mathbb{R}[X]$, define the set $S_n = \{p \in \mathbb{R}[X] \mid \#Z_p = n\}$, where Z_p is the zeroes set of p. Is it true that $\mathrm{Span}(S_n) = \mathbb{R}[X]$?

Exercise 7. Let N be a subspace of the space of linear maps $\mathscr{L}(V)$, where V is a finite-dimensional linear space. Suppose that in every pair of elements of N, two maps commute with each other, and every element in N is nilpotent. Prove that there exists $v \neq 0$ such that $f(v) = 0$ for every $f \in N$. [3]

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[1] This exercises was taken from the textbook "Problemi Scelti di Analisis Matematica I" authored by E. Acerbi, L. Modica and S. Spagnolo

[2] This exercises was taken from the final exam of the course of "Geometria I" of R. Benedetti, M. Ferrarotti and E. Fortuna on June 1997

[3] This exercise is a preliminary Lemma to the Engels' Theorem, P. Humpreys, "Introduction to Lie Algebras and Representation Theory"