SOLUTIONS OF EXERCISES OF WEEK ONE

Exercise 1. Let $\mathscr{A}$ be a non-empty collection of sets and $X \in \mathscr{A}$ be a set. Show that

$$\cap\mathscr{A} \subseteq X \subseteq \cup\mathscr{A}.$$

For both inclusions, you can start with the usual first step "Let $x \in \dots$".

Proof. Let x be an element of $\cap\mathscr{A}$. Then, for every $Y \in \mathscr{A}$, there holds $x \in Y$. In particular, $x \in X$. Now, let x be an element of X. Since $X \in \mathscr{A}$, there holds

$$x \in X \in \mathscr{A}$$

which means $x \in \cup\mathscr{A}$. □

Exercise 2. We checked that the set $\mathbf{N}_3 := \{1,2,3\}$ has exactly 24 choice functions. How many choice functions does the set $\mathbf{N}_4 := \{1,2,3,4\}$ have?

Proof. We have two choices for every pair, and there are $\binom{4}{2} = 6$ pairs. Then, we have three choices for every triple, and $\binom{4}{3} = 4$ triples. Finally, four choices for $\mathbf{N}_4$. Then, there are

$$2^6 \times 3^4 \times 4 = 20736$$

choice functions. □

Exercise 3. An equivalence relation xRy on A is just a subset $R \subseteq A \times A$ such that R is reflexive, symmetric and transitive. Let R be an equivalence relation on $\mathbf{N}_k := \{1,2,3,\dots,k\}$. Show that k is even if and only if $\#R$ is even.

Proof. We count R by dividing it into two subsets, the diagonal

$$D := \{(x,y) \in R \mid x = y\}$$

and its complement D^c. Then $\#R = \#D + \#(D^c)$. Since R is reflexive, $\#D = k$. Since R is symmetric, $\#(D^c)$ is an even number. Therefore, k is even if and only if $\#R$ is even. □

Exercise 4. Is it true that $2^{\cup\mathscr{B}} = \mathscr{B}$ for every non-empty collection of sets $\mathscr{B}$?

Proof. It is false. For instance consider $\mathscr{B} = \{\{0\},\{1\}\}$. Then

$$\cup\mathscr{B} = \{0,1\}, \quad 2^{\cup\mathscr{B}} = \{\emptyset,\{0\},\{1\},\{0,1\}\} \neq \mathscr{B}.$$

It is not difficult to find a lot of examples where this equality fails, especially in finite sets: in fact, if both $\mathscr{B}$ and $\cup\mathscr{B}$ are finite, then

$$\#2^{\cup\mathscr{B}} = 2^n, \quad (1,2,4,\dots)$$

while $\#\mathscr{B}$ is completely arbitrary. □

Exercise 5. Find the generalized unions and intersections of the following collections

(1) $$\mathscr{A}_1 := \{x\}$$

(2) $$\mathscr{A}_2 := \{[0,1+1/n) \mid n \geq 1\}$$

where $[a,b)$ is the interval of real numbers t such that $a \leq t < b$ and x is a set.

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Proof.

$$\cup\mathscr{A}_1 = x = \cap\mathscr{A}_1 = \{x\}.$$
$$\cup\mathscr{A}_2 = [0,2), \quad \cap\mathscr{A}_2 = [0,1].$$

□

SOLUTIONS OF EXERCISES OF WEEK TWO

Exercise 1. Let $A := \mathbf{R} - \mathbf{Z}$. Prove that A is dense in $\mathbf{R}$.

Proof. Given $a < b$ in $\mathbf{R}$, we consider two cases: $0 < b - a < 1$. Then $\mathbf{Z} \cap (a, b) = \emptyset$, therefore $x_* := (a + b)/2 \in A$ and $a < x_* < b$. On the other case, we have $b - a \geq 1$, then

$$a \leq b - 1 < b$$

and we choose $x_* = (2b - 1)/2$. □

Exercise 2 (*). When we consider the usual sum and multiplication in the complex field $\mathbb{C}$, the Field Axioms are satisfied. Prove that the Positive Set Axiom are not satisfied; in other words, prove that given a non-empty subset $P \subseteq \mathbb{C}$, at least one between P1) and P2) is false.

Proof. If P2) is true, then only one between $i \in P$ or $-i \in P$ (we ruled out $i = 0$) is true. If $i \in P$, then by P1), $-i = i^3 \in P$ and we obtain a contradiction. If $-i \in P$, then, by P1), $i = (-i)^3 \in P$ and we obtain another contradiction. □

Exercise 3. Given a set X and A a non-empty subset. Prove that the following

$$xRy \Leftrightarrow (x, y \in A) \vee (x, y \in A^c)$$

is an equivalence relation. What are the equivalence classes? In which cases is also an order relation?

Proof. We show that R is symmetric, transitive and reflexive.

(R) xRx means that either $x \in A$ or $x \in A^c$. This is true for every $x \in X$, because $X = A \cup A^c$

(S) if $x, y \in A$, clearly $y, x \in A$. The same applies to A^c

(T) suppose $xRy \wedge yRz$. Then xRz.

$$xRy \Rightarrow x, y \in A \vee x, y \in A^c.$$

$$yRz \Rightarrow y, z \in A \vee y, z \in A^c.$$

We have to check four different cases: firstly, we notice that the two cases

$$(x, y \in A) \wedge (y, z \in A^c), \quad (y, z \in A) \wedge (x, y \in A^c)$$

are not possible because both would imply $y \in A \cap A^c$. We discuss the remaining cases

$$(x, y \in A) \wedge (y, z \in A) \Rightarrow x, z \in A \Rightarrow xRz$$

$$(x, y \in A^c) \wedge (y, z \in A^c) \Rightarrow x, z \in A^c \Rightarrow xRz.$$

So, R is an equivalence relation. Now we address the questions whether R is an order relation. Since we already know that R is an order relation (so it is reflexive and transitive) we suppose that R is antisymmetric

(A) $xRy \wedge yRx \Rightarrow x = y.$

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We wish to draw some conclusions about the sets A and A^c. Given $x, y \in A$, we have
$$xRy \Rightarrow yRx$$
by (S). By (A),
$$(xRy \wedge yRx) \Rightarrow x = y.$$
Therefore,
$$x, y \in A \Rightarrow x = y.$$
Which means that A is a singleton. Now, suppose that $x, y \in A^c$. Similarly, we can show that $x = y$. The conclusion of this argument is: if R is an order relation, then A and A^c are singletons. Therefore,

(a) R is an equivalence relation
(b) if R is also an order relation, then A and A^c are singletons. And $\#X = 2$.

□

Exercise 4. Let $\mathscr{B}$ be a non-empty collection of sets. Prove or find a counterexample to each of the following statements:

(i) $\mathscr{B}$ is a finite set implies $\cup\mathscr{B}$ is a finite set
(ii) $\cup\mathscr{B}$ is a finite set implies $\mathscr{B}$ is a finite set.

Proof.

(i) It is not true: consider, for instance, $\mathscr{B} = \{\mathbf{N}\}$. The collection is finite, but $\cup\mathscr{B} = \mathbf{N}$ which is the set of natural numbers, not finite
(ii) it is true: in fact, we have
$$\mathscr{B} \subseteq 2^{\cup\mathscr{B}}.$$
Given $A \in \mathscr{B}$, there holds $A \subseteq \cup\mathscr{B}$. Then $A \in 2^{\cup\mathscr{B}}$; since $\cup\mathscr{B}$ is finite, its Power Set is finite, so $\mathscr{B}$ is finite (thanks to Hyeong-Jun for suggesting this solution).

□

EXERCISES OF WEEK THREE

Exercise 1 (ex. 16, page 16 of [1]). Prove that $\mathbf{Z}$ is countable.

Exercise 2. Write explicitly a Choice Function for the set of natural numbers $\mathbf{N}$ (do not use the Choice Axiom!).

Exercise 3. Prove that $\mathbf{R} \approx \mathbf{R} - \{0\}$.

Exercise 4. Given two natural numbers $h, k \geq 1$, use the induction principle to show that

(i) there exists $f\colon \mathbf{N}_h \to \mathbf{N}_k$ INJ $\Leftrightarrow h \leq k$
(ii) there exists $g\colon \mathbf{N}_h \to \mathbf{N}_k$ SURJ $\Leftrightarrow k \leq h$.

Exercise 5. Let A and B be two subsets of $\mathbf{R}$. Then

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}, \quad \overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Exercise 6. Let E be a closed set bounded from below. Then

$$\inf(E) \in E.$$

Exercise 7. Let E be a closed set such that

$$\inf(E \cap [a, +\infty)) = a$$

for every a real number. Then $E = \mathbf{R}$.

Exercise 8. Prove that if A is an open and closed set, then A is either $\emptyset$ or $\mathbf{R}$ (use Exercise 5).

References

1. P. M. Fitzpatrick and H. L. Royden, *Real analysis*, fourth ed., Pearson, 2010.

Date: 2016, March 20.

SOUTIONS OF EXERCISES OF WEEK FOUR

Exercise 1. Write explicitly a Choice Function for $\mathbf{Z}$. Write explicitly a Choice Function for the set of natural numbers $\mathbf{N}$ which is not $\phi(A) = \min(A)$ (and do not use the Choice Axiom!).

Proof. For $\mathbf{Z}$, we define

$$\phi(A) := \begin{cases} \min(A \cap \mathbf{N}) & \text{if } A \cap \mathbf{N} \neq \emptyset \\ -\min(-A \cap \mathbf{N}) & \text{if } A \cap \mathbf{N} = \emptyset. \end{cases}$$

For $\mathbf{N}$, we define

$$\psi(A) := \begin{cases} \min(A) & \text{if } A \neq \{1,2\} \\ 2 & \text{if } A = \{1,2\}. \end{cases}$$

□

Exercise 2. Prove that $[0,1] \approx [0,1)$.

Proof. We define

$$g(x) := \begin{cases} x & \text{if } x \neq \frac{1}{n} \text{ for every } n \in \mathbf{N} \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n}. \end{cases}$$

□

Exercise 3. Prove that a non-empty compact set E is is closed and bounded.

Proof. We prove that E is bounded. We consider the open cover

$$E \subseteq \cup \mathscr{U}, \quad \mathscr{U} := \{(-n,n) \mid n \geq 1\}.$$

Since E is compact, there exists a finite sub-cover $\mathscr{U}' \subseteq \mathscr{U}$. Since $\mathscr{U}'$ is finite, there exists n_0 such that

$$E \subseteq \mathscr{U}' = (-n_0, n_0).$$

We prove that E is closed. On the contrary, let x_0 be a point in $\overline{E} - E$. Since $x_0 \notin E$,

$$\mathscr{V} := \left\{ \left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n} \right) \mid n \geq 1 \right\}$$

is an open cover. Since E is compact, there exists a finite sub-cover $\mathscr{V}' \subseteq \mathscr{V}$. Then, there exists n_1 such that

$$E \subseteq \left(x_0 - \frac{1}{n_1}, x_0 + \frac{1}{n_1} \right)$$

□

Exercise 4. Let $\mathscr{B}$ be a finite σ-algebra. Show that $\#\mathscr{B}$ is even.

Proof. Since $\mathscr{B}$ is a σ-algebra, whenever $E \in \mathscr{B}$, the complement also belongs to $\mathscr{B}$. Since $E \neq E^c$, then $\#\mathscr{B}$ is even. □

Exercise 5 (Ex. 36, page 20 of [1])**.** Let J be the collection of the bounded intervals $[a,b)$ with $a < b$. Show that $\mathscr{B}(J) = \mathscr{B}(\Omega)$, the Borel's collection.

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Proof. The σ-algebra generated by a collection is obtained as the generalized intersection of

$$\mathscr{B}(A) := \cap \mathscr{F}_A.$$

Then, we have to prove that

$$\cap \mathscr{F}_\Omega = \cap \mathscr{F}_J.$$

Actually, we will prove that

$$\mathscr{F}_\Omega = \mathscr{F}_J.$$

Firstly, we show that $\mathscr{F}_\Omega \subseteq \mathscr{F}_J$. Let $\mathscr{B}$ a σ-algebra which contains Ω. We prove that $\mathscr{B} \supseteq J$. Let $[a, b)$ be in J. Then

$$[a, b) = \bigcap_{n=1}^{\infty} (a - 1/n, b).$$

Since $(a - 1/n, b) \in \Omega \subseteq \mathscr{B}$, $[a, b)$ is countable union of sets in $\mathscr{B}$. Then $[a, b) \in \mathscr{B}$. Now, let $\mathscr{B}$ be a σ-algebra which contains J. We prove that $\Omega \subseteq \mathscr{B}$. Let O be an open set. Then, there exists a countable collection of open intervals I_n such that

$$O = \bigcup_{n=1}^{\infty} I_n.$$

We prove that $I_n \in J$. In fact, if I_n is a bounded interval, we have $I_n = (a, b)$ and

$$(a, b) = \bigcup_{k=1}^{\infty} [a + 1/k, b)$$

while a similar expression holds for unbounded intervals. Since $[a + 1/k, b)$ is in $J \subseteq \mathscr{B}$, $(a, b) \in \mathscr{B}$. Then, O is countable union of sets of $\mathscr{B}$. Then $O \in \mathscr{B}$. ☐

References

1. P. M. Fitzpatrick and H. L. Royden, *Real analysis*, fourth ed., Pearson, 2010.