Infinite Graphs and Colorings of the Integers

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1 The main problem

We want to show that is possible to color the integers with a finite number of colors, say k, in order that, if

$$(b-a) = m!,$$

a and b have different colors.

2 Our attack

For some k, we will look for a magic constant $\eta \in \mathbb{R}$ such that the function

$$f_\eta : \mathbb{Z} \to \mathbb{R}/k\mathbb{Z}, \qquad f_\eta(n) = \eta \cdot n \pmod{k}$$

has the property that whenever $(b-a) = m!$ we have

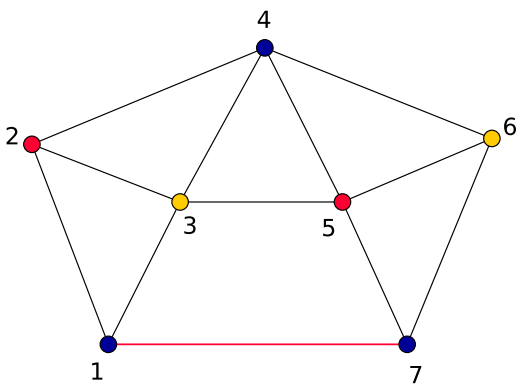
$$f_\eta(b) - f_\eta(a) \notin B(0,1), \tag{1}$$

where $B(0,1)$ is the open ball around 0 and of radius 1, with respect to the metric induced on $\mathbb{R}/k\mathbb{Z}$ by the standard metric on $\mathbb{R}$ (i.e. the smallest distance of two representatives). We will then define the coloring of n as $\lfloor \eta \cdot n \rfloor \pmod{k}$. Note that by linearity to fulfill the condition (1) it is sufficient to choose η such that:

$$f_\eta(m!) \notin B(0,1).$$

Theorem 1. *Let G be the infinite graph whose nodes are the integers, in which there is an arc between a and b if and only if $|b-a| = m!$. The chromatic number of G is at least 4.*

Proof.



□

Theorem 2. *The chromatic number of G is exactly 4.*

Proof. We will apply the construction outlined above taking

$$\eta = 2\cos(1) = \sum_{k=0}^{+\infty} \frac{2\cdot(-1)^k}{(2k)!}$$

and show that:

$$\forall m \geq 1, \quad f_\eta(m!) \notin B(0,1) \subset \mathbb{R}/4\mathbb{Z}.$$

We have:

$$\begin{cases} f_\eta(1) &= 1.0806\ldots \\ f_\eta(2) &= 2.1612\ldots \\ f_\eta(6) &= 2.4836\ldots \\ f_\eta(24) &= 1.9345\ldots \end{cases}$$

Now, for every $m > 4$,

$$f_\eta(m!) = \sum_{j=0}^{+\infty} \frac{2\cdot\chi_4(j+1)\cdot m!}{j!} \equiv \sum_{j=m-3}^{m} \frac{2\cdot\chi_4(j+1)\cdot m!}{j!} + \sum_{j>m} \frac{2\cdot\chi_4(j+1)\cdot m!}{j!} \pmod 4,$$

where χ_4 is the usual Dirichlet character:

$$\chi_4(r) = \begin{cases} 0 & \text{if} \quad r \equiv 0 \pmod 2 \\ 1 & \text{if} \quad r \equiv 1 \pmod 4 \\ -1 & \text{if} \quad r \equiv 3 \pmod 4 \end{cases}$$

and, by the integral form of the Taylor theorem:

$$\left| \sum_{j>m} \frac{2\cdot\chi_4(j+1)\cdot m!}{j!} \right| \leq 2\cdot \left| \int_0^1 (1-t)^m \frac{d^{m+1}}{dx^{m+1}}(\cos(x))\,dx \right| \leq \frac{2}{m+1} < \frac{2}{5}.$$

By defining:

$$g(m) = \sum_{j=m-3}^{m} \frac{2\cdot\chi_4(j+1)\cdot m!}{j!}$$

it is easy to check that:

$$\forall m \in \mathbb{N}, \quad g(m+1) - g(m) = 0, \tag{2}$$

so the function f_η with $\eta = 2\cdot\cos(1)$ maps the factorials into the ball $B(2, 0.92)$ in $\mathbb{R}/_{4\mathbb{Z}}$, prooving our claim. □

3 Generalization to arbitrary sequences of integers

In this section we will consider increasing integer sequences $(e_n)_{n\geq 1}$, and we will study the problem of determining if it is possible to color the integers with a finite amount of colors ensuring at the same time that each pair of integers a, b have different colors whenever $b - a = e_i$ for some $i \geq 1$.

Let's first note with a few examples that this problem seems not to be directly related to the density, or to the growth factor of the e_n. We have

Fact 1. *For each integers $N > 0$, there exist a sequence of density $1 - 1/N$ that allows a coloring of the integers with N different colors.*

Proof. In fact, for $N > 1$ take as e_n the set of integers that are $\neq 0 \pmod N$, and color the integers with N colors, depending on the class modulo N. □

Fact 2. *For each increasing function $f : \mathbb{N} \to \mathbb{N}$, there esist a sequence $(e_n)_{n\geq 1}$ satisfying $e_n > f(n)$, and which does not allow a coloring of the integers with a finite number of colors.*

Proof. Suppose by induction that there are integers $v_1, \ldots, v_m$, and that we defined $e_1, \ldots, e_n$, such that each $v_j - v_i$ for $1 \leq i \leq j \leq m$ is equal to some e_k, for $1 \leq k \leq n$. Now just put $v_{m+1} = e_n + v_m + f(n+m) + 1$, and put $e_{n+k} = v_{m+1} - v_{m+1-k}$, for $k = 1, \ldots, m$. It is now clear that the graph on $v_1, v_2, \ldots$ is the infinite complete graph, and it cannot be colored with a finite number of colors. □

However, density arguments are not completely unrelated to the problem:

Fact 3. *Let G be an infinite graph, having the positive integers as nodes, in which*

$$a - b \in \mathrm{Im}(f) \Longrightarrow \textit{there is an arc between } a \textit{ and } b.$$

Assume now that the chromatic number of G is finite, and call p_m^γ the $m-$th positive integer having color γ. If

$$\exists \varepsilon > 0 : \forall \gamma, \quad p_m^\gamma \gg m \log^{1+\varepsilon}(m),$$

G cannot have a finite chromatic number.

Proof. If we call C_γ the subset of the positive integers having color γ, we have:

$$\sum_{n=1}^{+\infty} \frac{1}{n} = \sum_{\gamma} \sum_{m \in C_\gamma} \frac{1}{m} \ll \sum_{\gamma} \sum_{m=2}^{+\infty} \frac{1}{m \log^{1+\varepsilon}(m)};$$

so, if the chromatic number of G is finite, the RHS is finite and the LHS is not. □

4 The Squares Conspiracy

We now ask if it is possible to color the positive integers with a finite number of colors, say k, in order that, if

$$(b - a) = m^2 > 0,$$

a and b have different colors.

Fact 4. *Let Q be $\{m^2 : m \in \mathbb{N} \setminus \{0\}\}$. There is a subset $R \subset Q$, having asymptotic density 1 in Q, that can be colored with a single color.*

Proof. It is sufficient to take

$$R = \{q^2 : q = a^2 + b^2 \text{ has no solutions}\}.$$

If (a, b, c) is a Pythagorean triple, $a^2 + b^2 = c^2$ and c is the sum of two squares, so we have:

$$(R - R) \cap Q = \emptyset,$$

and it is a well-known fact that $(Q + Q)$ has asymptotic density 0 in $\mathbb{N}$. Moreover, for every $n \in \mathbb{N}$,

$$n + R$$

can be colored with a single color. □

Fact 5. *Let G be the graph whose nodes are the positive integers, in which there is an arc between a and b if and only if $|a - b| = m^2 > 0$; let k be the chromatic number of G.*

$$k \geq 3.$$

Proof. Every Pythagorean triple gives a triangle in G, so $k > 2$. □

Fact 6. *There is no K_4 in G.*

Proof. There is a K_4 in G if and only if, for some $(a, b) \in \mathbb{N} \setminus \{0\}$ with $a \neq b$,

$$\begin{cases} a^2 + b^2 & \in & Q \\ a^2 - b^2 & \in & Q \end{cases}$$

Without loss of generality, we may assume $\gcd(a, b) = 1$ and $a > b$, so we have:

$$\begin{cases} a & = & 2uv \\ b & = & u^2 - v^2 \end{cases} \quad \text{or} \quad \begin{cases} a & = & u^2 - v^2 \\ b & = & 2uv \end{cases},$$

with $u > v > 0$, $\gcd(u, v) = 1$ and $uv \equiv 2 \pmod 4$. However, the first system of equations gives $a^2 - b^2 \equiv 3 \pmod 4$, so we have to solve:

$$\begin{cases} (u+v)^2 - 2v^2 & \in & Q \\ (u-v)^2 - 2v^2 & \in & Q \end{cases},$$

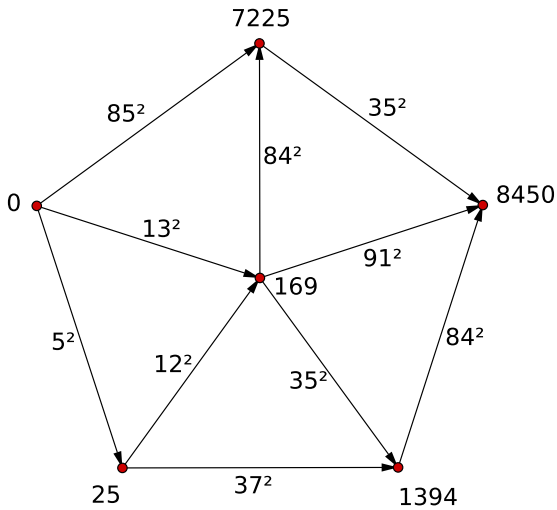
that is equivalent to the determination of a non-trivial rational point on the elliptic curve:

$$E(1) = \left\{(x, y) : y^2 = x^3 - x\right\}.$$

Fermat prooved [Fer], by the method of infinite descent, that $E(1)$ has no rational points apart from $\{(0, 0), (1, 0), (-1, 0)\}$, so there is no K_4 in G. □

Fact 7. *The chromatic number of G is greater than 3.*

Proof. There is a W_5 in G:



□

Fact 8. *There are many subsets $B \subset \mathbb{N}$ with asymptotic density 0 that can be colored with a single color.*

Proof. It is sufficient to take the set of the positive integers of the form:

$$N = \sum_{j=0}^{J} \delta_j \cdot 17^{2j} + \sum_{k=0}^{K} \epsilon_k \cdot 17^{2k+1},$$

with

$$\delta_j \in \Delta = \{3, 6, 13\},$$

$$\epsilon_k \in \{0, 1, \ldots, 15, 16\}.$$

The difference of two different elements of Δ is not a quadratic residue in $\mathbb{Z}/^*_{17\mathbb{Z}}$, so if:

$$(N_1 - N_2) \in Q \Longrightarrow 17^2 | (N_1 - N_2).$$

Furthermore, if we call B the set of such numbers, it is clear that both

$$B + m \quad \text{and} \quad m^2\, B$$

can be colored with a single color; the situation is the same for:

$$Q^2 = \{n^4 : n \geq 1\}.$$

□

Theorem 3 (Mycielski)**.** *For every k there is a triangle-free graph G with chromatic number k.*

Proof. C_5 is clearly the minimum triangle-free graph with chromatic number 3: we now proceed by induction. Given a triangle-free graph G with $\chi(G) = k$, let $V = \{v_1, \ldots, v_n\}$ the set of its vertices and $U(v_i)$ the open neighbourhood of v_i, i.e. the set of vertices v_j for which $v_i v_j$ is an edge in G. If we add $n+1$ vertices to G, $\{u_1, \ldots, u_n, w\}$, in order that:

$$\begin{cases} U(u_i) = U(v_1) \quad \forall i : 1 \leq i \leq n, \\ U(w) = \{u_1, \ldots, u_n\} \end{cases}$$

the chromatic number of the new graph G' is exactly $k+1$; moreover, G' is triangle-free. □

We say now that a (k-)coloring of $\mathbb{Z}$ is *linear* if the color of m is

$$\lfloor \alpha \cdot m \rfloor \pmod k$$

for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem 4. *There are no linear colorings of $\mathbb{Z}$ which are colorings for G.*

Proof. We simply state that, for every irrational number α,

$$\liminf_{n \in \mathbb{N}} \{n^2 \cdot \alpha\} = 0,$$

where

$$\{x\} = x - \lfloor x \rfloor.$$

Let

$$[a_0, a_1, \ldots, a_n, \ldots]$$

be the continued fraction of α, with convergents $\left\{\frac{p_0}{q_0}, \frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}, \ldots\right\}$. It is well known that:

$$\left\| \alpha - \frac{p_k}{q_k} \right\| < \frac{1}{q_k \cdot q_{k+1}} < \frac{1}{a_k \cdot q_k^2},$$

so, if the sequence $\{a_i\}_{i=0}^{+\infty}$ is unbounded (this is true for almost all irrational numbers),

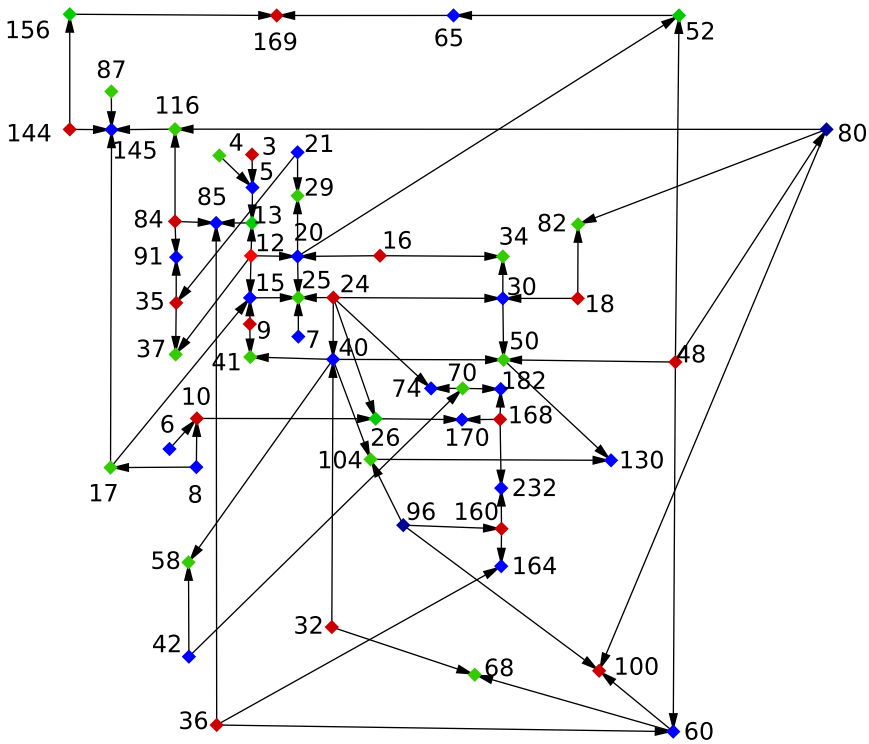
$$\liminf_{k \to +\infty} \{q_k^2 \cdot \alpha\} = 0.$$

But what about irrational numbers with bounded partial quotients? The following argument, communicated to us by Gowers [Gow], is due to Bergelson. Split $[0,1)$ into k equally sized sub-intervals, and color any natural number n according to which interval $\{\frac{n^2\alpha}{2}\}$ lies in. By van der Waerden's theorem we can find $n-d$, n and $n+d$ all of the same color. That implies that $\alpha \cdot \frac{(n-d)^2-2n^2+(n+d)^2}{2}$ is $\frac{1}{k}-$close to an integer. But it also equals $\alpha \cdot d^2$, so we are done (since k was arbitrary).
There is also a more effective argument due to Zaharescu [Zah]:

$$\|n^2 \cdot \alpha\| < \frac{1}{\sqrt{n}}$$

has infinitely many solutions. □

Let now $G_\square$ be the subgraph of G whose nodes belong to Q. Clearly, $G_\square$ is triangle-free, since G is K_4-free. If $G_\square$ were planar, in virtue of Grötzsch theorem it would have chromatic number equal to 3 (not less, since there is an infinite number of disjoint C_7 in $G_\square$).



However, $G_\square$ is not planar in virtue of Kuratowski theorem:

$$\text{We can go} \begin{cases} \text{from 84 to 40} & \text{through} & 91, 35, 29, 21, 20, 25, 24 \\ \text{from 84 to 60} & \text{through} & 85, 36 \\ \text{from 84 to 80} & \text{through} & 116 \\ \text{from 48 to 40} & \text{through} & 50 \\ \text{from 48 to 60} & & \\ \text{from 48 to 80} & & \\ \text{from 100 to 40} & \text{through} & 96, 104 \\ \text{from 100 to 60} & & \\ \text{from 100 to 80} & & \end{cases}.$$

Theorem 5 (Hindman)**.** *For every partition of $\mathbb{N}$ in a finite number of pieces, say $\mathbb{N} = B_1 \cup \ldots \cup B_r$, there is an infinite set $A \subset \mathbb{N}$ and an index $1 \le j \le r$ such that the finite sums of elements of A belong to B_j.*

We say that $S \subset \mathbb{N}$ is an IP-set if there is an infinite set A such that $\mathrm{FS}(A) \subset S$; we say that S is an IP*-set if $(\mathbb{N} \setminus S)$ is not an IP-set; we say that S is syndetic if its gap function is bounded, i.e.:

$$\sup_{s \in S} \left(\min\{t \in S : t > s\} - s\right) \in \mathbb{N}.$$

Fact 9. *If S is an IP-set, $(S - S)$ is syndetic.*

For a complete overview, see [Ber].

Theorem 6 (Jacobi triple product)**.**

$$\prod_{m=1}^{\infty} \left(1 - x^{2m}\right)\left(1 + x^{2m-1}y^2\right)\left(1 + x^{2m-1}y^{-2}\right) = \sum_{n=-\infty}^{\infty} x^{n^2} y^{2n}.$$

Fact 10. *In particular we have:*

$$\prod_{m=1}^{\infty} (1 - q^m) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}.$$

We say now that $A \subset \mathbb{N}$ is represented by a formal power series

$$f(z) = \sum_{n \in \mathbb{N}} a_n\, z^n, \quad a_n \in \mathbb{Z}$$

if

$$A = \{m \in \mathbb{N} : [z^m] f(z) > 0\}.$$

We have that for any finite coloring of $\mathbb{N}$, there is an infinite $A \subset \mathbb{N}$ such that

$$f(z) = \prod_{m \in A} (1 + z^m)$$

represents integers with the same color. By taking logarithmic derivatives on both sides, we have:

$$[z^m] f(z) = -\frac{1}{m} \sum_{j=1}^{m-1} \left([z^j] f(z)\right) \mu_A(m - j),$$

where:

$$\mu_A(n) = \sum_{\substack{d \mid n \\ d \in A}} d \cdot e^{\frac{\pi i n}{d}}.$$

We say now that the support of a formal power series

$$g(z) = \sum_{m \in \mathbb{Z}} g_m\, z^m, \quad g_m \in \mathbb{Z}$$

is the set

$$I_g = \{m \in \mathbb{Z} \setminus \{0\} : g_m \neq 0\}.$$

Under the assumption that G has a finite chromatic number, the supports of

$$g(z) = f(z) f(z^{-1}) = \prod_{m \in A} (1 + z^m)\left(1 + z^{-m}\right) \qquad \text{and} \qquad \theta(z) = \prod_{m=1}^{\infty} \left(1 - z^{2m}\right)\left(1 + z^{2m-1}\right)^2$$

must be disjoint: we say that g has to dodge f, meaning that $I_g \cap I_\theta$ must be the empty set.

Fact 11. *By assuming that G has a finite chromatic number, we can write $\mathbb{N}$ as a disjoint union of chromatic components $C_1, \ldots, C_k$. Without loss of generality, we can state that every C_i is unbounded and p-adically unbounded.*

Fact 12. *By assuming that G has a finite chromatic number, without loss of generality we can state that, for every chromatic component C_i, $|C_i \cap (m\mathbb{N}+n)| = \aleph_0$, for every $(m,n) \in (\mathbb{N}\setminus\{0\})^2$.*

Proof. Assume that G has chromatic number k and suppose that the first chromatic component, C_1, does not intersect $(m\mathbb{N}+n)$. We can assign to every natural number u a new color, i.e. the old color of $m^2 \cdot u + n$, still having a valid coloring. However, $(m^2\mathbb{N}+n)$ does not intersect C_1, so the procedure decrease the chromatic number of G, leading to a contradiction. In the case that the intersection between C_1 and $(m\mathbb{N}+n)$ is finite, we can apply a suitable backward-shift to $\mathbb{N}$ and fall into the previously analysed situation. □

Fact 13. *In the same hypothesis of the previous statements, in every chromatic component C_i, for every prime p, the p-adic norm ν_p is even infinitely many often.*

Proof. Every odd natural number is greater than 1, so if $\nu_p(C_i) \equiv 1 \pmod 2$, C_i does not intersect $(p\mathbb{N}+1)$. □

Theorem 7 (Erdös-Mordell)**.** *If μ is the Schnirelmann density, defined as*

$$\mu(A) = \inf_{n\geq 1} \frac{|A\cap[1,\ldots,n]|}{n},$$

then

$$\mu(A+Q) \geq \mu(A) + \frac{\mu(A)\,(1-\mu(A))}{8}.$$

Fact 14. *By assuming that G has a finite chromatic number, every chromatic component C_i satisfy:*

$$\mu(C_i) < \frac{1}{2}.$$

This is quite an obvious fact, since $\mu(C) \leq \frac{1}{\min C}$.

Fact 15. *For every chromatic component C_i we have:*

$$\mu\left((C_i+Q+Q)\cap C\right) \leq \mu\left((C-C)^+ \cap (Q+Q)\right) \leq \mu(Q+Q) = 0.$$

Analogously:

$$\mu\left((C_i+Q+Q+Q)\cap C\right) \leq \mu(Q+Q+Q) = \frac{5}{6}.$$

Fact 16. *Every chromatic component is syndetic.*

Proof. Suppose that $\chi(G) = k$ and C_1 has unbounded gaps: for every $L \in \mathbb{N}$ it is possible to find a coloring of $[1,\ldots,n]$ requiring only $(k-1)$ colors, corresponding to the backward-shift of the first gap in C_1 greater than L. □

Fact 17. *It is not restrictive to suppose that the maximum gap L_i appearing in the chromatic component C_i occurs infinitely many often.*

Fact 18. *Every chromatic component has a positive lower asymptotic density, not greater than $\frac{1}{2}$.*

Theorem 8 (Hindman)**.** *Every subset of $\mathbb{N}$ with positive upper Banach density is an IP-set.*

Let A_i be the infinite set whose finite sums belong to C_i. A_i has a null asymptotic density, otherwise, for some M,

$$\mathbb{N}\setminus[1,M] \subseteq \mathrm{FS}(A_i).$$

So $C_i \setminus A_i$ is still syndetic, then IP.

Fact 19. *For every chromatic component C_i*

$$\bigcup_{k=1}^{+\infty} \mathrm{FS}(A_{i,k}) \subseteq C_i$$

holds, where every $A_{i,k}$ is infinite and has a null asymptotic density.

Theorem 9. *Every natural number greater than* 50 *is the sum of exactly* 5 *positive squares.*

Proof. We use the Descartes conjecture (proved by Dubois in 1911, see [Bel]). The only natural numbers that cannot be represented as a sum of 4 positive squares are

$$1, 3, 5, 9, 11, 17, 29, 41 \quad \text{and all numbers of the form} \quad 2 \cdot 4^n, 6 \cdot 4^n, 14 \cdot 4^n,$$

so, if $m > 50$, al least one number between $(m-1)$ and $(m-9)$ can be represented as a sum of 4 positive squares: this is equivalent to the statement that the quinary diagonal quadratic form $\sum_{j=1}^{5} x_j^2$ is almost universal. As a corollary, we have also that every square number q^2, with $q > 3$, can be represented as a sum of exactly 4 positive squares, giving a C_5 in G. □

Theorem 10 (Sárközy)**.** *Any subset $A \subset \{1, \ldots, N\}$ which contains no square difference has density*

$$\alpha \ll \frac{(\log \log N)^{2/3}}{(\log N)^{1/3}}.$$

This powerful theorem [Sar], developed through the Hardy-Littlewood circle method, implies that there are no syndetic sets whose differences avoid the squares.

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